قضیه نوتر:

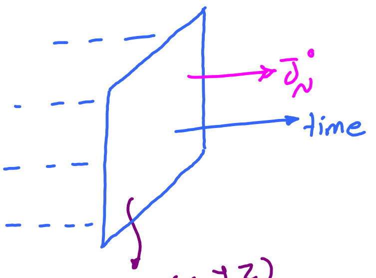
تبدیل کلی: $\varphi(x) \longrightarrow \varphi(x) + \delta\varphi(x)$

بینهایت: $\delta\varphi(x) = D\varphi\,\delta\lambda$ ①

$\delta S = 0$: تحت این تبدیل، کنش ناورداست

$$J^{\mu}_{N} = \Pi^{\mu}(x)\, D\varphi(x) - W^{\mu}(x) \quad ; \quad D\mathcal{L} = \partial_\mu W^{\mu} \text{ ②} \longrightarrow \partial^{\mu} J_{\mu}^{N} = 0$$

جریان نوتر پایسته است.



$$\frac{d}{dt}\int d^3x\, J_N^0 = 0$$

بار نوتر: Q_N

انتقال فضا-زمان

$$x^\mu \longrightarrow x'^\mu = x^\mu + \delta x^\mu$$

میدانها تابع فضا-زمان هستند. پس میدانها هم تبدیل می‌یابند.

$$①,② : \quad \delta\mathcal{L} = \underbrace{D\mathcal{L}}_{\partial_\mu W^\mu}\,\delta\lambda = (\partial_\mu W^\mu)\,\delta\lambda \qquad \partial_\mu(a^\mu \varphi) \quad \Longrightarrow W^\mu = a^\mu \mathcal{L}$$

همچنین:

$$\delta\varphi = \frac{\partial\varphi}{\partial\lambda}\Big|_{\lambda=0}\delta\lambda = D\varphi\,\delta\lambda = (a^\mu \partial_\mu \varphi)\,\delta\lambda$$

$$\varphi(x^\mu) \longrightarrow \varphi(x^\mu + \lambda a^\mu)$$

تانسور انرژی-تکانه: $T^\mu{}_\nu$

$$J^\mu_N = \Pi^\mu \underbrace{D\varphi}_{a^\nu \partial_\nu \varphi} - W^\mu = \Pi^\mu a^\nu \partial_\nu \varphi - \delta^\mu_\nu a^\nu \mathcal{L} = a^\nu \big[\underbrace{\Pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}}_{T^\mu{}_\nu}\big]$$

پایسته: $J^0_N \ (\mu = 0) \Longrightarrow T^0{}_\nu \longrightarrow P^\nu = \int d^3x\, T^{0\nu}$ (*) $\longrightarrow P^0 = \int d^3x\, T^{00}$

$$\Longrightarrow P^0 = \int d^3x \left[\Pi^0 \partial_0 \varphi - \mathcal{L}\right] = \int d^3x \big[\underbrace{\Pi(x)\dot\varphi(x) - \mathcal{L}}_{\mathcal{H}}\big] \Longrightarrow P^0 = \int d^3x\, \mathcal{H} \ : \text{انرژی}$$

چگالی هامیلتونی: $\mathcal{H}$

$$* : \nu = k : \quad P^k = \int d^3x\, T^{0k} = \int d^3x \left[\Pi^0 \partial_k \varphi - \cancel{\delta^0_k}\mathcal{L}\right] = \int d^3x\, \Pi(x)\,\partial^k \varphi(x) \ . \ \text{تکانه}$$

اگر سیستم تحت انتقال فضا-زمان ناوردا باشد، پایستگی انرژی-تکانه داریم.

فصل ۱۱: کوانتش کانونیک میدانها

$$\varphi(\vec{x},t): \text{classical field} \xrightarrow[\text{کوانتش کانونیک}]{\text{ماشین}} \hat\varphi(x) : \text{Quantum field}$$

روش کوانتش کانونیک

۱) چگالی لاگرانژی کلاسیک برحسب میدان‌ها می‌نویسیم.

۲) چگالی تکانه مزدوج را بدست می‌آوریم و چگالی هامیلتونی را برحسب میدان‌ها می‌نویسیم.

۳) به میدان و چگالی تکانه مزدوج به صورت اپراتوری نگاه می‌کنیم. روابط جابجایی بین آنها را می‌نویسیم.

۴) میدان‌ها را برحسب اپراتورهای خلق و فنا بسط می‌دهیم $\longleftarrow$ occupation number

۵) Normal ordering را وارد کنید.

Massive scalar field

1) $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 \longrightarrow$ معادله حرکت : $(\Box + m^2)\varphi = 0$

2) $\Pi^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi \Longrightarrow \Pi^0(x) = \Pi(x) = \partial^0 \varphi(x) = \dfrac{\partial \varphi(x)}{\partial t} = \dot{\varphi}(x)$

$$\mathcal{H} = \Pi^0 \partial_0 \varphi(x) - \mathcal{L} = \underbrace{\partial^0 \varphi(x)\, \partial_0 \varphi(x)}_{(\partial_0 \varphi)^2} - \Big\{ \underbrace{\tfrac{1}{2}(\partial_0 \varphi)^2}_{\text{قسمت زمانی}} - \underbrace{\tfrac{1}{2}(\vec{\nabla}\varphi)^2}_{\text{قسمت فضایی}} \Big\} + \tfrac{1}{2} m^2 \varphi^2$$

$$\Longrightarrow \mathcal{H} = \underbrace{\tfrac{1}{2}(\partial_0 \varphi)^2}_{\text{انرژی جنبشی (تغییرات زمانی میدان)}} + \underbrace{\tfrac{1}{2}(\vec{\nabla}\varphi)^2}_{\text{shear term : تغییرات فضایی}} + \underbrace{\tfrac{1}{2} m^2 \varphi^2}_{\text{پتانسیل}}$$

3) field $\longrightarrow$ field operator

$\varphi(x) \longrightarrow \hat{\varphi}(x)$

$\Pi^0(x) \longrightarrow \hat{\Pi}^0(x)$

$\mathcal{H} \longrightarrow \hat{\mathcal{H}}$

داشتیم : $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \longrightarrow \underbrace{[\hat{\varphi}(t,\vec{x}), \hat{\Pi}^0(t,\vec{y})]}_{\text{equal-time commutator}} = i\,\delta^{(3)}(\vec{x}-\vec{y})$

$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \longrightarrow [\hat{\varphi}(x), \hat{\varphi}(y)] = [\Pi^0(x), \Pi^0(y)] = 0$

داشتیم : $x_j = \sqrt{\dfrac{\hbar}{m}} \sum\limits_k \dfrac{1}{\sqrt{2\omega_k N}} \left(\hat{a}_k e^{ijka} + \hat{a}^\dagger_k e^{-ijka}\right)$ ؛ $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'}$

4) $\hat{\varphi}(\vec{x}) = \displaystyle\int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(\hat{a}_p e^{i\vec{p}\cdot\vec{x}} + \hat{a}^\dagger_p e^{-i\vec{p}\cdot\vec{x}}\right)$ ؛ $E_p = \mp\sqrt{\vec{p}^2 + m^2}$

$[\hat{a}_p, \hat{a}^\dagger_{p'}] = \delta^{(3)}(\vec{p} - \vec{p}')$

در تصویر هایزنبرگ :

$$\hat{\varphi}(t,\vec{x}) = \hat{\varphi}(x) = U^{\dagger}(t,0)\,\hat{\varphi}(\vec{x})\,U(t,0) = e^{i\hat{H}t}\,\hat{\varphi}(\vec{x})\,e^{-i\hat{H}t}$$

$$\longrightarrow \hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \Big(\underbrace{U^{\dagger}\hat{a}_p}_{e^{-iE_pt}\hat{a}_p} e^{i\vec{p}\cdot\vec{x}}\, U + \underbrace{U^{\dagger}a^{\dagger}_p}_{e^{+iE_pt}\hat{a}^{\dagger}_p} e^{-i\vec{p}\cdot\vec{x}}\, U \Big)$$

$p = (E, \vec{p})$

$x = (t, \vec{x})$

$p\cdot x = Et - \vec{p}\cdot\vec{x}$

9.40 رابطه : $\hat{U}^{\dagger}(t,0)\,\hat{a}_p\,\hat{U}(t,0) = e^{-iE_pt}\,\hat{a}_p$

$\qquad\qquad a^{\dagger}_p \qquad\qquad = e^{+iE_pt}\,\hat{a}^{\dagger}_p$

$$\Longrightarrow \boxed{\hat{\varphi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \left(\hat{a}_p e^{-ip\cdot x} + \hat{a}^{\dagger}_p e^{+ip\cdot x} \right)} \rightsquigarrow \hat{\varphi}^{\dagger}(x) = \hat{\varphi}(x)$$

بررسی بیشتر

$H|n\rangle = E_n|n\rangle$

$e^{iHt}|n\rangle = e^{iE_nt}|n\rangle$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$e^{iHt}\,\hat{a}_p\,e^{i\vec{p}\cdot\vec{x}}\,\underbrace{e^{-iHt}\,|n_{\vec{p}}\;n_{\vec{q}}\;n_{\vec{r}}\rangle}$$

$|n_{\vec{p}}\;n_{\vec{q}}\;n_{\vec{r}}\rangle\, e^{-i(n_pE_p + n_qE_q + n_rE_r)t}$

$\sqrt{n_p}\,|(n_p-1)\;n_q\;n_r\rangle$

$e^{i([n_p-1]E_p + n_qE_q + n_rE_r)t}\,|n_p-1\;n_q\;n_r\rangle$

$$= \sqrt{n_p}\,e^{i\vec{p}\cdot\vec{x}}\,e^{-iE_pt}\,|n_p-1\;n_q\;n_r\rangle = \underbrace{\sqrt{n_p}\,e^{-ip\cdot x}\,|n_p-1\;n_q\;n_r\rangle}_{\hat{a}_p|n_p\;n_q\;n_r\rangle}$$

$$\longrightarrow U^{\dagger}\hat{a}_p e^{i\vec{p}\cdot\vec{x}}U = \hat{a}_p e^{-ip\cdot x}$$

Normalizing factors

$\int d^3p$: ناوردای لورنتس نیست $\longleftarrow$ فضای چهاربعدی

$$\downarrow$$

$$d^4p \longrightarrow d^4p\,\underbrace{\delta(p^2-m^2)\,\Theta(p^0)}_{\frac{1}{2E_p}\delta(p^0-E_p)\Theta(p^0)}$$

$\Theta(p^0)$: $\underset{\sim}{\;} E>0$

$p^{\mu} = (E, \vec{p}) \longrightarrow p^{\mu}p_{\mu} = E^2 - \vec{p}^{\,2} = m^2$

$\longrightarrow p^2 = m^2$: mass shell condition

شرط ذرات فیزیکی است.

ثابت می‌کنیم : $\delta(p^2-m^2)\,\Theta(p^0) = \frac{1}{2E_p}\,\delta(p_0-E_p)\,\Theta(p^0)$

یادآوری : $\delta[f(x)] = \sum_{\{x_0|f(x_0)=0\}} \frac{1}{|f'(x)|_{x=x_0}}\,\delta(x-x_0)$

$x = p^0 \quad , f(p^0) = p^2 - m^2 = (p^0)^2 - \vec{p}^{\,2} - m^2 \Longrightarrow |f'(p^0)| = 2|p^0|$

$\hookrightarrow f(p^0) = 0 \longrightarrow p^0 = \mp\sqrt{\vec{p}^{\,2} - m^2} = \mp E_p$

$$\Longrightarrow \delta(p^2-m^2)\,\theta(p_0)=\left\{\frac{1}{2|p^0|}\Big|_{p^0=E_p}\delta(p^0-E_p)+\frac{1}{2|p^0|}\Big|_{p^0=-E_p}\cancel{\underbrace{\delta(p^0+E_p)}_{p^0=-E_p}}\right\}\theta(p_0)$$

$$\longrightarrow \delta(p^2-m^2)\,\theta(p_0)=\frac{1}{2E_p}\delta(p^0-E_p)\,\theta(p_0)$$

پس measure انتگرال‌گیری در فضای ۴ بعدی
برای حالت‌های فیزیکی به این صورت است:

$$\int d^3p\,\frac{1}{2E_p}\frac{1}{(2\pi)^3}=\int \overbrace{d^3p\,dp^0}^{d^4p}\,\frac{1}{2E_p}\,\delta(p^0-E_p)\,\underbrace{\theta(p^0)}_{\text{انرژی + است}}$$

$$1=\int d^3p\,|\vec{p}\rangle\langle\vec{p}| \longrightarrow 1=\int\frac{d^3p}{(2\pi)^3\,2E_p}|p\rangle\langle p|$$

$$\langle\vec{p}|\vec{q}\rangle=\delta^{(3)}(\vec{p}-\vec{q}) \longrightarrow \langle p|q\rangle=(2\pi)^3(2E_p)\,\delta^{(3)}(\vec{p}-\vec{q})$$

حالت ۴ مستوی: $|p\rangle=(2\pi)^{3/2}\sqrt{2E_p}\,|\vec{p}\rangle$

همچنین اپراتورها را نیز به همین صورت به شکل هموردای لورنتس در می‌آوریم :

$$\hat{a}^{\dagger}_p=(2\pi)^{3/2}\sqrt{2E_p}\,a^{\dagger}_p \rightsquigarrow \hat{a}^{\dagger}_p|0\rangle=(2\pi)^{3/2}\sqrt{2E_p}\,\underbrace{a^{\dagger}_p|0\rangle}_{|\vec{p}\rangle}=|p\rangle$$

$$\longrightarrow \hat{\varphi}(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\Big(\underbrace{\hat{a}_p}_{(2\pi)^{3/2}\sqrt{2E_p}\,a_p}e^{-ip\cdot x}+\hat{a}^{\dagger}_p e^{ip\cdot x}\Big)$$

$$\longrightarrow \varphi(x)=\int\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x}+a^{\dagger}_p e^{+ip\cdot x}\right)^{(I)}\ \checkmark$$

هامیلتونی

$$H=\int \mathcal{H}\,d^3x=\frac{1}{2}\int d^3x\left\{(\partial_0\varphi)^2+(\vec{\nabla}\varphi)^2+m^2\varphi^2\right\}$$

$$\hat{\Pi}_\mu=\partial_\mu\hat{\varphi}(x)\overset{(I)}{=}\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\left[-ip_\mu\,a_p e^{-ip\cdot x}+(+ip_\mu)\,a^{\dagger}_p e^{ip\cdot x}\right]$$

$$\mu=0\longrightarrow \partial_0\varphi(x)=\int\frac{d^3p}{----}\,(-i\underbrace{p_0}_{=E_p})\left[a_p e^{-ip\cdot x}-a^{\dagger}_p e^{+ip\cdot x}\right]$$

$$\mu=k\longrightarrow -\vec{\nabla}\varphi(x)=\int ----\,(-i\vec{p})\left[\quad\text{''}\quad\text{''}\quad\right]$$

$$H=\frac{1}{2}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^3\sqrt{2E_p}\sqrt{2E_q}}\Big\{\underbrace{(-iE_p)(-iE_q)}_{-E_pE_q}\left(a_p e^{-ip\cdot x}-a^{\dagger}_p e^{+ip\cdot x}\right)\left(a_q e^{-iq\cdot x}-a^{\dagger}_q e^{iq\cdot x}\right)$$

$$+\underbrace{(i\vec{p})\cdot(i\vec{q})}_{-\vec{p}\cdot\vec{q}}\,(\quad {}^{\prime\prime}\quad)(\quad {}^{\prime\prime}\quad) + m^2\left(\hat{a}_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)\left(\hat{a}_q e^{-iq\cdot x} + a_q^\dagger e^{iq\cdot x}\right)\Big\}$$

$$= \frac{1}{2}\int \frac{d^3p\, d^3q}{\cancel{(2\pi)^3}\, 2\sqrt{E_p E_q}} \Big\{ \cancel{(2\pi)^3}\, \delta^{(3)}(\vec{p}-\vec{q}) \xrightarrow{\vec{p}\to\vec{q}} \left[E_p E_q + \vec{p}\cdot\vec{q} + m^2\right] \left(a_p a_q^\dagger e^{-i(E_p-E_q)t}\right.$$

$$\left. + a_p^\dagger a_q e^{-i(E_p-E_q)t}\right) + \cancel{(2\pi)^3}\, \delta^{(3)}(\vec{p}+\vec{q}) \xrightarrow{\vec{p}\to-\vec{q}} \left[-E_p E_q - \vec{p}\cdot\vec{q} + m^2\right]\left(a_p^\dagger a_q^\dagger e^{-i(E_p+E_q)t}\right.$$

$$\left. + a_p a_q e^{-i(E_p+E_q)t}\right)\Big\}$$

$$\int d^3x\, e^{-ip\cdot x} = \underbrace{\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}}_{(2\pi)^3\,\delta(\vec{p})} e^{iEt}$$

$$= \frac{1}{2}\int \frac{d^3p}{\cancel{2E_p}} \Big\{ (\underbrace{E_p^2 + \vec{p}^2 + m^2}_{\cancel{2E_p^2}})(a_p^\dagger a_p + a_p a_p^\dagger)$$

$$(\underbrace{-E_p^2 + \vec{p}^2 + m^2}_{0})\left(a_p^\dagger a_{-p}^\dagger e^{2iE_pt} + a_p a_{-p} e^{-2iE_pt}\right)\Big\}$$

$$\Longrightarrow H = \frac{1}{2}\int d^3p\, E_p(\underbrace{a_p a_p^\dagger}_{a_p^\dagger a_p + \delta^3(0)} + a_p^\dagger a_p) = \int d^3p\, E_p\left(a_p^\dagger a_p + \underline{\tfrac{1}{2}\delta^{(3)}(0)}\right) \quad \text{II}$$

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(3)}(\vec{p}-\vec{q})$$

Normal ordering

$$N(ABC^\dagger X^\dagger YZ) = C^\dagger X^\dagger ABYZ \quad : \quad \text{برای بوزون‌ها}$$

$$N[aa^\dagger] = a^\dagger a \qquad N[a^\dagger a] = a^\dagger a$$

$$N[a^\dagger a a a^\dagger a^\dagger] = a^\dagger a^\dagger a^\dagger a a$$

$$N[\underbrace{cc^\dagger}_{-c^\dagger c} c] = -c^\dagger c c$$

نوسانگرهای هماهنگ ساده : $H = \sum_k \hbar\omega_k N_k$

اپراتور تعداد : $\underline{N_p}$

$$N[H] = \frac{1}{2}\int d^3x\, E_p\left\{\underbrace{N(a_p a_p^\dagger)}_{a_p^\dagger a_p} + \underbrace{N(a_p^\dagger a_p)}_{a_p^\dagger a_p}\right\} \Longrightarrow H = \int d^3x\, E_p\, a_p^\dagger a_p$$

* مسائل 11.1 و 11.2 را حل کنید.