جلسه هفتم: فرمول‌های رایلی، رفتار توابع ($x\to 0$ و $x\to\infty$)، تعامد توابع بسل

$$j_0(x)=\frac{\sin x}{x}\ ,\quad n_0(x)=-\frac{\cos x}{x}\ ,\quad h_0^{(1)}=\frac{-ie^{-ix}}{x}\ ,\quad h_0^{(2)}=\frac{ie^{-ix}}{x}$$

$$j_\nu(x)=\sqrt{\frac{\pi}{2x}}\,J_{\nu+\frac12}(x)\xrightarrow{\nu=0} j_0(x)=\sqrt{\frac{\pi}{2x}}\,J_{\frac12}(x)=\sqrt{\frac{\pi}{2x}}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\underbrace{\Gamma(r+\frac12+1)}_{\frac{(2r+2)!}{2^{2r+2}(r+1)!}\sqrt{\pi}}}\left(\frac{x}{2}\right)^{2r+\frac12}$$

$$\Longrightarrow \cdots\ j_0(x)=\frac{1}{x}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}=\frac{\sin x}{x}$$

برای n_0: حل به عهده دانشجو

$$n_\nu(x)=\sqrt{\frac{\pi}{2x}}\,N_{\nu+\frac12}(x)\xrightarrow{\nu=} n_0(x)=\sqrt{\frac{\pi}{2x}}\,N_{\frac12}(x)$$

داریم

$$N_\nu(x)=\frac{J_\nu(x)\cos\nu\pi-J_{-\nu}(x)}{\sin\nu\pi}\xrightarrow{\nu=\frac12}\cdots\ ,\quad \Gamma\left(-\tfrac12+r+1\right)=\Gamma\left(r+\tfrac12\right)=\frac{(2r)!}{2^{2r}r!}\sqrt{\pi}$$

$$\Longrightarrow\cdots\Longrightarrow n_0(x)=-\frac1x\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{(2r)!}=-\frac1x\cos x$$

فرمول‌های رایلی

اگر ν عدد صحیح باشد، داریم: $\nu=n$

1) $j_n(x)=(-1)^n x^n\left(\frac1x\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$

2) $n_n(x)=-(-1)^n x^n\left(\frac1x\frac{d}{dx}\right)^n\left(\frac{\cos x}{x}\right)$

3) $h_n^{(1)}(x)=-i(-1)^n x^n\left(\frac1x\frac{d}{dx}\right)^n\left(\frac{e^{ix}}{x}\right)$

4) $h_n^{(2)}(x)=i(-1)^n x^n\left(\frac1x\frac{d}{dx}\right)^n\left(\frac{e^{-ix}}{x}\right)$

اثبات 1: روش استقرا

$n=0$: $j_0(x)=(-1)^0x^0\left(\frac1x\frac{d}{dx}\right)^0\frac{\sin x}{x}=\frac{\sin x}{x}$ ✓

فرض: رابطه 1 برای $n=N$ درست است و اثبات می‌کنیم برای حالت $n=N+1$ هم درست است.

فرض: $j_N(x)=(-1)^N x^N\left(\frac1x\frac{d}{dx}\right)^N\left(\frac{\sin x}{x}\right)$

حکم: $j_{N+1}(x)=(-1)^{N+1}x^{N+1}\left(\frac1x\frac{d}{dx}\right)^{N+1}\left(\frac{\sin x}{x}\right)$?

روابط بازگشتی: $\frac{d}{dx}\left(x^{-\nu}f_\nu(x)\right)=-x^{-\nu}f_{\nu+1}(x)\ \xRightarrow[\nu=N]{}\ j_{N+1}(x)=-x^N\frac{d}{dx}\left(x^{-N}j_N(x)\right)$

f_ν: $j_\nu,\ n_\nu,\ h_\nu^{(1)},\ h_\nu^{(2)}$

$$\Longrightarrow j_{N+1}(x) = -x^{N} \frac{d}{dx}\left(\cancel{x^{-N}} (-1)^{N} \cancel{x^{N}} \left(\frac{1}{x}\frac{d}{dx}\right)^{N} \frac{\sin x}{x}\right)$$

$$= (-1)^{N+1} x^{N} \frac{d}{dx}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^{N} \frac{\sin x}{x}\right] \times \frac{x}{x}$$

$$= (-1)^{N+1} x^{N+1} \underbrace{\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)^{N}} \frac{\sin x}{x} = (-1)^{N} x^{N+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{N+1} \frac{\sin x}{x} \checkmark$$

$$j_1(x) = (-1)^1 x^1 \left(\frac{1}{x}\frac{d}{dx}\right)\frac{\sin x}{x} = -\frac{d}{dx}\left(\frac{\sin x}{x}\right) = -\frac{x\cos x - \sin x}{x^2} = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$n_1(x) = \cdots = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$h_1^{(1)} = j_1(x) + i\,n_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} + i\left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right) = \frac{-i}{x^2}(\cos x + i\sin x)$$

$$-\frac{1}{x}(\cos x + i\sin x) = -\frac{i}{x^2} e^{ix} - \frac{1}{x} e^{ix} = -i\frac{e^{ix}}{x}\left(\frac{1}{x} - i\right)$$

رفتار مجانبی توابع

قضیه: برای $x \to \infty$ داریم:

1) $J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left\{x - (\nu + \tfrac{1}{2})\frac{\pi}{2}\right\}$

2) $N_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \sin\left\{x - (\nu + \tfrac{1}{2})\pi/2\right\}$

3) $H_\nu^{(1)}(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \exp\left\{i\left(x - (\nu + \tfrac{1}{2})\pi/2\right)\right\}$

4) $H_\nu^{(2)}(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \exp\left\{-i\left(x - (\nu + \tfrac{1}{2})\right)\pi/2\right\}$

5) $I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^{x}$

6) $K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}$

7) $j_\nu(x) \sim \frac{1}{x}\sin(x - \nu\pi/2)$

8) $n_\nu(x) \sim \frac{-1}{x}\cos(x - \nu\pi/2)$

9) $h_\nu^{(1)} \sim \frac{-i}{x}\exp[i(x - \nu\pi/2)]$

10) $h_\nu^{(2)} \sim \frac{i}{x}\exp[-i(x - \nu\pi/2)]$

اثبات رابطه ۹ :

حل بیش : $K_\nu(x) = \dfrac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\dfrac{x}{2}\right)^\nu \displaystyle\int_1^\infty e^{-xt}\,(t^2-1)^{\nu-\frac{1}{2}}\,dt$

تغییر متغیر : $t = 1+\dfrac{u}{x} \longrightarrow dt = \frac{1}{x}du \;,\; xt = x+u \;,\quad t=1 \longrightarrow u=0 \;,\; t=\infty \longrightarrow u=\infty$

$$K_\nu(x) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-(x+u)}\Big[\underbrace{\cancel{1}+u^2/x^2+\frac{2u}{x}-\cancel{1}}_{\left[2u/x\left\{\frac{u}{2x}+1\right\}\right]^{\nu-\frac{1}{2}}}\Big]^{\nu-\frac{1}{2}}\frac{1}{x}\,du$$

$$= \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(x/2\right)^\nu \frac{e^{-x}}{x}\left(\frac{2}{x}\right)^{\nu-\frac{1}{2}}\int_0^\infty u^{\nu-\frac{1}{2}}\Big[\cancel{\frac{u}{2x}}+1\Big]^{\nu-\frac{1}{2}}e^{-u}\,du$$

$\xrightarrow{x\to\infty} 0$

انتگرال $= \displaystyle\int_0^\infty e^{-u}\,u^{\overset{(\nu+\frac{1}{2})-1}{\nu-\frac{1}{2}}}\,du = \Gamma(\nu+\tfrac{1}{2})$ یادآوری : $\Gamma(z) = \displaystyle\int_0^\infty e^{-u}\,u^{z-1}\,du$

$$\Longrightarrow K_\nu(x\to\infty) = \frac{\sqrt{\pi}}{\cancel{\Gamma(\nu+\frac{1}{2})}}\,\frac{e^{-x}}{x}\left(\frac{2}{x}\right)^{-\frac{1}{2}}\cancel{\Gamma(\nu+\tfrac{1}{2})} = \sqrt{\pi/2}\,\frac{e^{-x}\cancel{\sqrt{x}}}{\cancel{x}\,\sqrt{x}} = \sqrt{\frac{\pi}{2x}}\,e^{-x} \xrightarrow{x\to\infty} 0$$

اثبات رابطه ۳ :

داریم : $K_\nu(x) = \dfrac{\pi}{2}\,i^{\nu+1}H_\nu^{(1)}(ix)$

$$x \longrightarrow -ix \Longrightarrow H_\nu^{(1)}\big(\underbrace{i(-ix)}_{x}\big) = \frac{2}{\pi}K_\nu(-ix)\,\underbrace{i^{-\nu-1}}_{(e^{i\pi/2})^{-\nu-1} = e^{-i\pi/2(\nu+1)}} \qquad e^{i\pi/2}=i$$

از بالا جایگذاری می‌کنیم : $\sqrt{\dfrac{\pi}{2(-ix)}}\,e^{-(-ix)}$

$$\Longrightarrow H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\,e^{ix}\,e^{-i\pi/2(\nu+1)} = \sqrt{\frac{2}{\pi x}}\,e^{i[x-(\nu+1)\pi/2]} \quad \checkmark \xrightarrow{x\to\infty} 0$$

اثبات رابطه ۴ :

$$H_\nu^{(2)} = H_\nu^{(1)*} \sim \sqrt{\frac{2}{\pi x}}\,e^{-i[x-(\nu+1)\pi/2]} \quad \checkmark \xrightarrow{x\to\infty} 0$$

اثبات ۱ و ۲ :

$$H_\nu^{(1)} = J_\nu + iN_\nu \longrightarrow J_\nu(x) = \mathrm{Re}\,H_\nu^{(1)} \;,\; N_\nu(x) = \mathrm{Im}\,H_\nu^{(1)}(x)$$

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left[x-(\nu+1)\pi/2\right] + i\sin\left[x-(\nu+\tfrac{1}{2})\pi/2\right]\right\}$$

$$\longrightarrow J_\nu(x) = \mathrm{Re}\,H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\cos\left[x-(\nu+\tfrac{1}{2})\pi/2\right] \;*$$

$$\longrightarrow N_\nu(x) = \operatorname{Im} H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \sin\left[x-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}\right]$$

اثبات رابطه ۵ :

داشتیم: $I_\nu(x) = i^{-\nu} J_\nu(ix)$ $\qquad i = e^{i\pi/2}$

$$\overset{*}{\longrightarrow} \xrightarrow{x \to ix} J_\nu(ix) = \sqrt{\frac{2}{\pi \underbrace{i}_{\to e^{i\pi/2}} x}} \cos\left[ix-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}\right] \qquad \cos\alpha = \frac{1}{2}\left(e^{i\alpha}+e^{-i\alpha}\right)$$

$$\Rightarrow I_\nu(x) = e^{-i\pi/4} e^{-i\frac{\pi}{2}\nu} \sqrt{\frac{\cancel{2}}{\pi x}}\ \frac{1}{\cancel{2}_{\sqrt{2}}} \left\{ \underbrace{e^{i\left[ix-(\nu+\frac{1}{2})\frac{\pi}{2}\right]}}_{e^{-x} e^{-i\nu\pi/2} e^{-\pi/4 i}} + \underbrace{e^{-i\left[ix-(\nu+\frac{1}{2})\frac{\pi}{2}\right]}}_{e^{x} e^{i\nu\pi/2} e^{i\pi/4}} \right\}$$

$0 \qquad x \to \infty$

$$I_\nu(x) = \frac{1}{\sqrt{2\pi x}} e^{x} \quad \checkmark$$

اثبات رابطه ۷:

داشتیم: $j_\nu(x) = \sqrt{\frac{\pi}{2x}} J_{\nu+\frac{1}{2}}(x) \underset{*}{=} \sqrt{\frac{\cancel{\pi}}{\cancel{2}x}} \sqrt{\frac{\cancel{2}}{\cancel{\pi}x}} \cos\left[x-(\nu+\tfrac{1}{2}+\tfrac{1}{2})\tfrac{\pi}{2}\right]$

$* : \nu \to \nu+\frac{1}{2}$

$$\longrightarrow j_\nu(x) \sim \frac{1}{x}\cos\left[x-(\nu+1)\tfrac{\pi}{2}\right] = \frac{1}{x}\cos\left[-\tfrac{\pi}{2}+x-\nu\tfrac{\pi}{2}\right] = \frac{1}{x}\sin(x-\nu\tfrac{\pi}{2}) \quad \checkmark$$

اثبات رابطه ۸ :

داشتیم : $n_\nu(x) = \sqrt{\frac{\pi}{2x}} N_{\nu+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi x}} \sin\left[x-(\nu+1)\tfrac{\pi}{2}\right]$

$$= \frac{-1}{x}\cos\left[x-\nu\tfrac{\pi}{2}\right] \quad \checkmark$$

اثبات رابطه ۹:

$$h_\nu^{(1)} = j_\nu(x) + i n_\nu(x) = \frac{1}{x}\sin(x-\nu\tfrac{\pi}{2}) - \frac{i}{x}\cos(x-\nu\tfrac{\pi}{2})$$

$$= -i/x\left[\cos(x-\nu\tfrac{\pi}{2}) + i\sin(x-\nu\tfrac{\pi}{2})\right] = -i/x\ e^{i(x-\nu\frac{\pi}{2})} \quad \checkmark$$

اثبات رابطه ۱۰:

$$h_\nu^{(2)} = h_\nu^{(1)*} = \frac{i}{x} e^{-i(x-\nu\frac{\pi}{2})} \quad \checkmark$$

رفتار توابع در x های کوچک

توضیح: وقتی $x \to 0$ داریم:

$$\text{I)}\quad J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}$$

$$\text{II)}\quad N_\nu(x) \sim \begin{cases} -\frac{1}{\pi}\Gamma(\nu)\left(\frac{2}{x}\right)^{\nu} & \text{if } \nu \neq 0 \\ \frac{2}{\pi}\ln x & \text{if } \nu = 0 \end{cases}$$

III) $j_n(x) \sim \dfrac{x^n}{(2n+1)!!}$ $\qquad n:\text{integer}$

IV) $n_n(x) \sim \dfrac{(2n-1)!!}{x^{n+1}}$ $\qquad n:\text{integer}$

اثبات I :

چون $x \ll 1$ ، توان‌های کوچکتر مهم‌تر هستند.

داریم : $$J_\nu(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(\nu+r+1)} \left(\frac{x}{2}\right)^{2r+\nu}$$

$$r=0 \Longrightarrow J_\nu(x\to 0) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \checkmark \xrightarrow{x\to 0} 0$$

اثبات II :

داریم : $$N_\nu(x) = \frac{J_\nu \cos\nu\pi - J_{-\nu}}{\sin\nu\pi} \xrightarrow{x\to 0} N_\nu(x\to 0) \sim \frac{-1}{\sin\nu\pi} J_{-\nu}(x)$$

$$J_{-\nu}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-\nu+r+1)}\left(\frac{x}{2}\right)^{2r-\nu} \xrightarrow[\text{پایین‌ترین توان}]{r=0} J_{-\nu}(x) \sim \frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu}$$

$$\Longrightarrow N_\nu(x\to 0) \sim \frac{-1}{\sin\nu\pi\,\Gamma(-\nu+1)}\left(\frac{2}{x}\right)^\nu \sim \frac{-\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu \checkmark$$

$$\Gamma(1-\nu)\Gamma(\nu) = \frac{\pi}{\sin\nu\pi}$$

قرار بود به عنوان تمرین شکل $N_n(x)$ را بدست آورید :

11.60 رابطه : $N_0(x) = \frac{2}{\pi}(\ln x + \gamma - \ln 2) + O(x^2) \longrightarrow N_\nu(x) \sim \frac{2}{\pi}\ln x \,;\, \nu = 0$ $\checkmark$

اثبات III :

$$j_\nu(x) = \sqrt{\frac{\pi}{2x}}\, J_{\nu+1/2}(x) \underset{\text{از رابطه I}}{=} \sqrt{\frac{\pi}{2x}}\, \frac{1}{\Gamma(\nu+1/2+1)}\left(\frac{x}{2}\right)^{\nu+1/2}$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$(\nu+1/2)\,\Gamma(\nu+1/2) = (\nu+1/2)\,\Gamma(\nu-1/2+1) \qquad (\nu-1/2)\,\Gamma(\nu-1/2)$$

$$j_\nu(x) = \sqrt{\frac{\cancel{\pi}}{2x}}\left(\frac{x}{2}\right)^{\nu+1/2} \frac{1}{\underbrace{(\nu+1/2)(\nu-1/2)\cdots \tfrac{3}{2}\,\tfrac{1}{2}\,\Gamma(1/2)}_{\nu+1\ \text{جمله}}} = \frac{1}{2^{1/2}\cancel{x^{1/2}}}\, \frac{x^{\nu+\cancel{1/2}}}{2^{\nu+\cancel{1/2}}}\, \frac{2^{\cancel{\nu+1}}}{(2\nu+1)(2\nu-1)\cdots 1}$$

$\Gamma(1/2) = \sqrt{\pi}$

ν عدد صحیح است.

مثال : $\nu = 2 :$ $$\Gamma(2+1/2+1) = (2+1/2)\,\Gamma(2+1/2) = (2+1/2)(2-1/2)\,\Gamma(2-1/2) = \underbrace{(2+1/2)(2-1/2)(1-1/2)}_{\text{۳ جمله}}\,\Gamma(1/2)$$

$5/2 \quad 3/2 \quad 1/2$

$\Gamma(2-1/2) = \Gamma(1-1/2+1)$

$$\longrightarrow j_\nu(x) \sim \frac{x^\nu}{(2\nu+1)(2\nu-1)\cdots 3\cdot 1} = \frac{x^\nu}{(2\nu+1)!!}$$

اثبات رابطه IV:

$$n_n(x) = \sqrt{\frac{\pi}{2x}}\, N_{n+1/2}(x) \underset{\text{از رابطه II}}{=} \sqrt{\frac{\pi}{2x}} \left\{ -\frac{1}{\pi} \Gamma(n+1/2) \left(\frac{2}{x}\right)^{n+1/2} \right\}$$

$$\Gamma(n-1/2+1) = (n-1/2)\underbrace{\Gamma(n-1/2)}_{\Gamma(n-3/2+1)} = (n-1/2)(n-3/2)\Gamma(n-3/2)$$

$$\Rightarrow n_n(x) \sim -\sqrt{\frac{\cancel{\pi}}{2x}}\, \frac{1}{\cancel{\pi}} \underbrace{(n-1/2)(n-3/2)\cdots 3/2 \cdot 1/2}_{\text{n تا جمله}} \cdot \underbrace{\Gamma(1/2)}_{\cancel{\sqrt{\pi}}} \left(\frac{2}{x}\right)^{n+1/2}$$

$$= -\sqrt{\frac{1}{\cancel{2}x}}\, \frac{1}{\cancel{2^n}} \underbrace{(2n-1)(2n-3)\cdots 3\cdot 1}_{(2n-1)!!} \frac{2^{\cancel{n+1/2}}}{x^{n+1/2}} = \frac{-(2n-1)!!}{x^{n+1}} \checkmark$$

* تمرین: روابط زیر را اثبات کنید، $x \to 0$

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} (x/2)^\nu$$

$$\underset{\nu\neq 0}{K_\nu(x)} \sim \frac{\Gamma(\nu)}{x^\nu} 2^{\nu-1} \quad , \quad K_0(x) \sim -\ln x$$

$H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x) \longrightarrow$ روابط 11.88 تا 11.91

$$h_\nu^{(1)},\ h_\nu^{(2)} \longrightarrow \mp i\, n_\nu(x)$$

تعامد توابع بسل

معادله بسل: $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$

معادله خودالحاقی نیست $\Rightarrow P_0' = 2x \neq P_1$

شکل کلی معادله دیفرانسیل: $P_0 y'' + P_1 y' + P_2 y = 0 \Longrightarrow P_0 = x^2,\ P_1 = x,\ P_2 = x^2 - \nu^2$

شرط خودالحاقی بودن معادله دیفرانسیل: $P_1 = P_0'$

اگر معادله خودالحاقی نباشد، می توان آن را با ضرب در فاکتور $\frac{f(x)}{P_0}$ خودالحاقی کرد که $f'(x) = \frac{f P_1}{P_0}$.

برای معادله بسل: $f'(x) = \frac{f \cancel{x}}{x^{\cancel{2}}} \to \frac{df}{dx} = \frac{f}{x} \Rightarrow \frac{df}{f} = \frac{dx}{x} \to \ln f = \ln x \to f = x$

پس فاکتور: $\frac{f(x)}{P_0} = \frac{x}{x^2} = \frac{1}{x}$ معادله را خودالحاقی می کند.

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \xrightarrow[\times 1/x]{} x y'' + y' + \frac{1}{x}(x^2 - \nu^2) y = 0$$

$$P_0 = x \ ,\ P_1 = 1 \Longrightarrow P_0' = 1 = P_1 \checkmark$$