جلسه دوم: نمادگذاری فاکتوریل دوگانه   تابع‌ی گاما و بتا گاما   مبحث آوردن انتگرال‌های معین

$$\beta(p,q) \equiv \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad \operatorname{Re} p>0 \ , \operatorname{Re} q>0$$

اثبات کنید: $\beta(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$$\Gamma(z)=\int_0^\infty e^{-t}\,t^{z-1}\,dt \qquad \operatorname{Re} z>0$$

$$\Gamma(p)\Gamma(q)=\int_0^\infty e^{-t}\,t^{p-1}\,dt\int_0^\infty e^{-s}\,s^{q-1}\,ds=\int_0^\infty\int_0^\infty e^{-(t+s)}\,t^{p-1}s^{q-1}\,dt\,ds$$

$(t,s)\longrightarrow(u,z)$

تغییر متغیر: $t=uz \ , \ s=z(1-u) \Longrightarrow t+s=z$

$$u=\frac{t}{z}=\frac{t}{t+s} \ : \ \int_0^\infty dt \longrightarrow \int_0^1 du \ , \ \int_0^\infty ds \longrightarrow \int_0^\infty dz$$

ژاکوبین Jacobian: $f:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ ، ژاکوبین تبدیل یک ماتریس $n\times m$ است:

$$J\equiv\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m}\\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m}\end{pmatrix}$$

مثال: $(r,\theta)\longrightarrow(x,y)$

$x=r\cos\theta$

$y=r\sin\theta$

$f:\mathbb{R}^+\times[0,2\pi]\longrightarrow\mathbb{R}^2$

$$J=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\longrightarrow|J|=r$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}g(x,y)\,dx\,dy=\int_0^\infty\int_0^{2\pi}G(r,\theta)\,r\,dr\,d\theta$$

$$|J|=\begin{vmatrix}\frac{\partial t}{\partial u} & \frac{\partial t}{\partial z}\\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial z}\end{vmatrix}$$

برمی‌گردیم به مسأله خودمان:

$s=z(1-u)\longrightarrow\dfrac{\partial s}{\partial u}=-z \ , \ \dfrac{\partial s}{\partial z}=1-u$

$t=uz\longrightarrow\dfrac{\partial t}{\partial z}=u \ , \ \dfrac{\partial t}{\partial u}=z$

$$\longrightarrow |J|=\begin{vmatrix}z & u\\ -z & 1-u\end{vmatrix}=z(1-u)+uz=z\,.$$

$$\longrightarrow \Gamma(p)\Gamma(q)=\int_0^\infty\int_0^1 e^{-z}(uz)^{p-1}\left[z(1-u)\right]^{q-1}\underbrace{|J|}_{z}\,du\,dz$$

$$= \underbrace{\int_0^\infty e^{-z} z^{p+q-1}\, dz}_{\Gamma(p+q)} \underbrace{\int_0^1 u^{p-1}(1-u)^{q-1}\, du}_{\beta(p,q)} \Longrightarrow \Gamma(p)\Gamma(q) = \Gamma(p+q)\,\beta(p,q)$$

نشان دهید $\underbrace{\Gamma(z+1) = z\Gamma(z)}_{Re\, z>0}$ قابل تعمیم به zهای منفی غیرصحیح است.

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \quad ①$$

$$\lim_{z\to 0^+} \Gamma(z) = \lim_{z\to 0^+} \frac{\Gamma(z+1)=1}{z} = \frac{1}{0^+} = +\infty$$

$\longrightarrow$ $\Gamma(0)$ نامشخص است $\Longrightarrow \Gamma(-m) \to \infty$

m: integer

$$\lim_{z\to 0^-} \Gamma(z) = \lim_{z\to 0^-} \frac{\Gamma(1)=1}{z} = \frac{1}{0^-} = -\infty$$

سمت چپ رابطه ① برای $Re\, z>0$ و سمت راست آن برای $Re\, z>-1$ برقرار است.

با تکرار این رابطه می‌توان سمت چپ را برای $Re\, z>-1$ و سمت راست را برای $Re\, z>-2$ نوشت. پس رابطه ① برای تمامی اعداد منفی غیرصحیح درست است (توجه کنید مثلاً $\Gamma(\frac{1}{2})=\sqrt{\pi} \longrightarrow \Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$)

قراردادی فاکتوریل دوگانه

$$1\cdot 3\cdot 5 \cdots \equiv (2n+1)!!$$

$$(-1)!! \equiv 1$$

$$2\cdot 4\cdot 6 \cdots = (2n)!!$$

$$2\cdot 4\cdot 6 \cdots = 2(1)\cdot 2(2)\cdot 2(3)\cdots 2(n) = 2^n(1\cdot 2\cdot 3\cdots n) = 2^n n! \Longrightarrow (2n)!! = 2^n n!$$

$$1\cdot 3\cdot 5 \cdots = \frac{1\cdot 2\cdot 3\cdot 4\cdots}{2\cdot 4\cdot 6\cdots} = \frac{(2n+1)!}{2^n n!} \longrightarrow (2n+1)!! = \frac{(2n+1)!}{2^n n!}$$

تابع دیگاما Digamma

داشتیم: $\Gamma(z+1) = z\Gamma(z)$ ، $\Gamma(z+1) = z!$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!}{z(z+1)\cdots(z+n)} n^z \longrightarrow \Gamma(z+1) = z! = \lim_{n\to\infty} \frac{n!}{(z+1)\cdots(z+n)} n^z$$

$$\to \underbrace{\ln(z!)}_{\ln\Gamma(z+1)} = \lim_{n\to\infty} \left[\ln n! - \ln(z+1) - \cdots - \ln(z+n) + z\ln n\right]$$

از طرفین رابطه مشتق می‌گیریم

تابع دیگاما : $\psi(z+1) \equiv \frac{d \ln\Gamma(z+1)}{dz} = \frac{d}{dz} \ln(z!) \longrightarrow \psi(z) = \frac{d \ln\Gamma(z)}{dz} = \frac{d}{dz} \ln(z-1)!$

$$\Rightarrow \psi(z+1) = \lim_{n\to\infty} \left[-\frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n} + \ln n \right] + \left(1+\tfrac{1}{2}+\cdots \tfrac{1}{n}\right) - \left(1+\tfrac{1}{2}+\cdots\tfrac{1}{n}\right)$$

[illegible] : $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$

$$\Rightarrow \psi(z+1) = -\gamma + \lim_{n\to\infty} \left\{ 1 - \frac{1}{z+1} + \frac{1}{2} - \frac{1}{z+2} + \cdots + \frac{1}{n} - \frac{1}{z+n} \right\}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n}\right) = \sum_{n=1}^{\infty} \frac{z}{n(z+n)}$$

$$\Rightarrow \boxed{\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}}$$

polygamma توابع پلی‌گاما

$$z=0 \longrightarrow \psi(1) = -\gamma$$

$$\frac{d\psi(z+1)}{dz} = \psi^{(1)}(z+1) = \frac{d^2}{dz^2} \ln\Gamma(z+1) = \sum_{n=1}^{\infty} \frac{n(\cancel{z}+n) - n\cancel{z}}{n^2 (z+n)^2} = \sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$$

$$\frac{d^2\psi(z+1)}{dz^2} = \psi^{(2)}(z+1) = \frac{d^3}{dz^3} \ln\Gamma(z+1) = \sum_{n=1}^{\infty} \frac{-2}{(z+n)^3} = (-1)^{2+1}\, 2! \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2+1}}$$

$$\Rightarrow \frac{d^m\psi(z+1)}{dz^m} = \boxed{\psi^{(m)}(z+1) = \frac{d^{m+1}}{dz^{m+1}} \ln\Gamma(z+1) = (-1)^{m+1}\, m! \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}} \quad m=1,2,\ldots}$$

تابع پلی‌گاما

$$\psi^{(m)}(1) = (-1)^m\, m! \sum_{n=1}^{\infty} \frac{1}{n^{m+1}}$$

$z=0$

$$\zeta(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^x} : \begin{cases} \text{if } x>1 : \text{تابع همگراست} \\ \text{if } x\leq 1 : \text{واگراست} \end{cases}$$

تابع زتای ریمان

$$\psi^{(m)}(1) = (-1)^m\, m!\, \zeta(m+1)$$

تمرین 8.2.6 :

$$\psi^{(m)}(z+2) = \psi^{(m)}(z+1) + (-1)^m \frac{m!}{(z+1)^{m+1}} \qquad m=0,1,2,\ldots$$

انتگرال های معین برحسب توابع گاما

$$\int_0^\infty \frac{t^{p-1}\,dt}{(bt+a)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{a^q b^p\,\Gamma(p+q)}$$

شکل دوم تابع β : $\beta(p,q) = \int_0^\infty \frac{s^{p-1}}{(1+s)^{p+q}}\,ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

تغییر متغیر : $s = b/a\; t \Rightarrow ds = b/a\; dt$

$$\Rightarrow \beta(p,q) = \int_0^\infty \frac{(b/a)^{p-1}\,t^{p-1}}{(1+b/a\;t)^{p+q}}\,b/a\;dt = \int_0^\infty \frac{\cancel{a^{-p+1}}\,b^{p-1}\,t^{p-1}}{\cancel{a^{-p-q}}\,(a+bt)^{p+q}}\,b\cancel{a^{-1}}dt$$

$$= a^q b^p \int_0^\infty \frac{t^{p-1}}{(a+bt)^{p+q}}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \checkmark$$

نتیجه فرعی

$$t = \tan^2\theta \;\Big\downarrow\; dt = 2\tan\theta\,\frac{1}{\cos^2\theta}\,d\theta$$

$$\to \int_0^{\pi/2} \frac{\tan^{2p-\cancel{2}\,-1}\theta}{(a+b\tan^2\theta)^{p+q}}\,2\cancel{\tan\theta}\,\frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{a^q b^p}\,\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$L.H.S = \int_0^{\pi/2} \frac{2\sin^{2p-1}\theta\,\cos^{\cancel{2p}+2q}\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)^{p+q}\,\cos^{\cancel{2p}-1}\theta\,\cos^2\theta}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin^{2p-1}\theta\,\cos^{2q-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{p+q}}\,d\theta = \frac{1}{2a^q b^p}\,\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

حالتهای خاص

$$a=b=1 \;,\; 2p-1=p' \;,\; 2q-1=q' \longrightarrow p=\tfrac{1}{2}(p'+1) \;,\; q=\tfrac{1}{2}(q'+1)$$

$$\Rightarrow \int_0^{\pi/2} \sin^{p'}\theta\,\cos^{q'}\theta\,d\theta = \frac{1}{2}\,\frac{\Gamma\left(\frac{p'+1}{2}\right)\Gamma\left(\frac{q'+1}{2}\right)}{\Gamma\left(\frac{p'+q'+2}{2}\right)}$$

$$q'=0 \xrightarrow[p'=n]{} \int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \quad \sqrt{\pi} \qquad (4)$$

$$p'=q'=n \longrightarrow \int_0^{\pi/2} \sin^n\theta \cos^n\theta \, d\theta = \frac{1}{2} \frac{\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2}{\Gamma(n+1)}$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^p}} = \frac{\Gamma(1/p)}{\Gamma(1/p+1/2)} \frac{\sqrt{\pi}}{p}$$

تغییر متغیر: $t^p = \sin^2\theta \Longrightarrow t = \sin^{2/p}\theta \longrightarrow dt = 2/p \sin^{2/p-1}\theta \cos\theta \, d\theta$

$$\Longrightarrow \int_0^{\pi/2} \frac{2/p \sin^{2/p-1}\theta \cancel{\cos\theta} \, d\theta}{\cos\theta} = 2/p \int_0^{\pi/2} \sin^{2/p-1}\theta \, d\theta \underset{(4)}{=} \frac{\cancel{2}}{p} \times \frac{\sqrt{\pi}}{\cancel{2}} \frac{\Gamma(1/p)}{\Gamma(1/p+1/2)}$$

$n = 2/p - 1$

$$\int_0^{\pi/2} \frac{d\theta}{\sin\theta} \int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta = \pi$$

$$(4) \longrightarrow n=1/2: \int_0^{\pi/2} d\theta \sin^{1/2}\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(3/4)}{\Gamma(5/4)} = I_1$$

$$n=-1/2: \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(1/4)}{\Gamma(3/4)} = I_2$$

$$\Longrightarrow I_1 \times I_2 = \frac{\pi}{4} \frac{\Gamma(\cancel{1/4})}{\Gamma(5/4)} = \pi \checkmark$$

$\frac{1}{4}\Gamma(\cancel{1/4})$

داشتیم: $\Gamma(z+1) = z\Gamma(z) \xrightarrow{z=1/4} \Gamma(1+1/4) = 1/4 \, \Gamma(1/4)$

مثال: معادله حرکت یک ذره از حالت سکون در حضور نیروی جاذب $\frac{-mk}{x}$ در فاصله $[0,a]$ به صورت زیر داده شده است:

از نقطه a

ثابت کنید: $T = a\sqrt{\frac{\pi}{k}}$ (دوره تناوب)

$$F = m\frac{d^2x}{dt^2} \Longrightarrow \frac{d^2x}{dt^2} + \frac{k}{x} = 0$$

$$\frac{d}{dx}\frac{dx}{dt} \longleftarrow \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dx}\left(\frac{dx}{dt}\right)^2$$

$$\xrightarrow{\times dx} d\left(dx/dt\right)^2 = -k/x \, dx \longrightarrow \left(\frac{dx}{dt}\right)^2 = -k \ln x + C$$

v

شرایط : $x = a \;:\; v = 0 \Longrightarrow 0 = -k\ln a + C \longrightarrow C = k\ln a$

$$\Longrightarrow v^2 = -k\ln x + k\ln a = k\ln\frac{a}{x} \longrightarrow v = \frac{dx}{dt} = \underbrace{\sqrt{k}}_{\alpha}\sqrt{\ln a/x}$$

$$\Longrightarrow \int_0^a \frac{dx}{\alpha\sqrt{\ln a/x}} = \int_0^T dt \longrightarrow \frac{1}{\alpha}\int_0^a \left(\ln a/x\right)^{-1/2} dx = T$$

$x = a \equiv y = 0 \;,\; x = 0 \equiv y = \infty$

تغییر متغیر : $y = \ln a/x = -\ln x/a \longrightarrow e^{-y} = +x/a$

$dx = -a e^{-y} dy \longleftarrow x = +a e^{-y}$

$$\longrightarrow \frac{+1}{\alpha}\int_{\cancel{\infty}\,0}^{\cancel{0}\,\infty} y^{-1/2}(+a)e^{-y} dy = T \Longrightarrow T = \frac{a}{\sqrt{k}}\underbrace{\int_0^\infty y^{-1/2} e^{-y} dy}_{\Gamma(1/2) = \sqrt{\pi}} = a\sqrt{\frac{\pi}{k}}$$