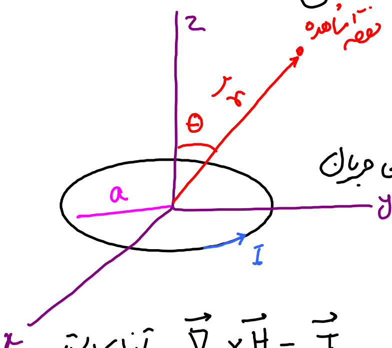
جلسه نوزدهم: حل مثال   هماهنگ‌های کروی   تقارن زاویه‌ای و عملگرهای نردبانی

$$\int_{-1}^{+1} P_l^m P_{l'}^m \, dx = \frac{2}{2l+1}\,\frac{(l+m)!}{(l-m)!}\,\delta_{ll'}$$

$$\int_{-1}^{+1} P_l^m(x)\, P_l^{m'}(x)\, \underbrace{(1-x^2)^{-1}}_{\text{تابع وزن}} \, dx = \frac{(l+m)!}{m(l-m)!}\,\delta_{mm'}$$



مثال: میدان مغناطیسی و پتانسیل برداری ناشی از حلقه دایره‌ای حامل جریان I را در نقطه‌ای دلخواه از فضا برای $r>a$ محاسبه کنید.

حجمی جریان $\vec{J} = J\,\hat{\phi}$ ؛ $\vec{A} = A(r,\theta)\,\hat{\phi}$: از تقارن مسئله

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

قانون آمپر $\vec{\nabla}\times\vec{H} = \vec{J}$ ؛ $\vec{H} = \vec{B}/\mu_0 \longrightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$

$$\underbrace{\vec{\nabla}\times\vec{A}}_{(I)}^{=\vec{B}} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\, A(r,\theta) \end{vmatrix} = \frac{1}{r^2\sin\theta}\left\{ \hat{r}\,\frac{\partial}{\partial\theta}(r\sin\theta\, A) - r\hat{\theta}\,\frac{\partial}{\partial r}(r\sin\theta\, A)\right\}$$

$$\vec{\nabla}\times\vec{\nabla}\times A = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta A) & \frac{-\cancel{r^2}}{\cancel{r^2}\sin\theta}\frac{\partial}{\partial r}(r\sin\theta A) & 0 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \left[-\frac{\partial^2 A}{\partial r^2} - \frac{2}{r}\frac{\partial A}{\partial r} - \frac{1}{r^2}\frac{\partial^2 A}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial}{\partial\theta}(\cot\theta\, A)\right]\frac{\hat{\phi}}{r\sin\theta} = \mu_0\vec{J} = \mu_0 J\hat{\phi}$$

$$\partial^2 A/\partial r^2 + 2/r\;\partial A/\partial r + 1/r^2\;\partial^2 A/\partial\theta^2 + 1/r^2\;\frac{\partial}{\partial\theta}(\cot\theta\, A) = 0 \longleftarrow \vec{J} = 0$$: خارج حلقه

$$A(r,\theta) = R(r)\,P(\theta)$$

$$\underbrace{-\frac{1}{\sin^2\theta}RP + R\cot\theta\frac{dP}{d\theta}}$$

$$P\frac{d^2R}{dr^2} + P\frac{2}{r}\frac{dR}{dr} + \frac{1}{r^2}R\frac{d^2P}{d\theta^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}(\cot\theta\, RP) = 0$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r^{\cancel{2}}}{R}\frac{2}{\cancel{r}}\frac{dR}{dr} = \frac{-1}{P}\,d^2P/d\theta^2 + \frac{1}{\sin^2\theta} - \frac{1}{P}\cot\theta\frac{dP}{d\theta} = K$$

$$\Longrightarrow \begin{cases} r^2 \dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr} - \cancel{k}^{\,l(l+1)} R = 0 \longrightarrow R(r) = A r^{l} + B r^{-(l+1)} \\[2ex] \dfrac{d^2P}{d\theta^2} + \cot\theta \dfrac{dP}{d\theta} + \left[\cancel{k}^{\,l(l+1)} - \dfrac{1}{\sin^2\theta}\right] P = 0 \end{cases}$$

$*$: لژاندر وابسته $(m=1) \longrightarrow P = P_l^1(\cos\theta)$

لژاندر وابسته : $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + \left[l(l+1) - \dfrac{m^2}{1-x^2}\right]y = 0$

$$x = \cos\theta \longrightarrow 1-x^2 = \sin^2\theta \quad , \quad \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy}{d\theta}\frac{1}{-\sin\theta}$$

$$dx = -\sin\theta\, d\theta \qquad \frac{d^2y}{dx^2} = \frac{d\theta}{dx}\frac{d}{d\theta}\left[\frac{-1}{\sin\theta}\frac{dy}{d\theta}\right] = +\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{dy}{d\theta}\right]$$

$$\Rightarrow \sin^{\cancel{2}}\theta \left\{ \frac{1}{\cancel{\sin\theta}}\left[\frac{-\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} + \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right]\right\} - 2\cos\theta\frac{1}{-\sin\theta}\frac{dy}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0$$

$$\Longrightarrow \cancel{-\cot\theta\frac{dy}{d\theta}} + \frac{d^2y}{d\theta^2} + \cancel{2}\cot\theta\frac{dy}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0 \rightsquigarrow$$

که دقیقاً رابطه $*$ است با $m=1$

$$\longrightarrow A(r,\theta) = \sum_{l=1}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l^1(\cos\theta) = \sum_{l=1}^{\infty} \underbrace{C_l}_{?} \left(\frac{a}{r}\right)^{l+1} P_l^1(\cos\theta)$$

$A_l = 0$ برای این که تابع بینهایت واگرا نشود.

I از رابطه : $B_r = (\vec{\nabla}\times\vec{A})_r = \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta A) = \dfrac{1}{r\sin\theta}\cos\theta A + \dfrac{1}{r\cancel{\sin\theta}}\cancel{\sin\theta}\dfrac{\partial A}{\partial\theta}$ $*$

$$\sum_{l=1}^{\infty} C_l \left(\frac{a}{r}\right)^{l+1} \frac{dP_l^1}{d\theta} \; ?$$

بازگشتی : $(1-x^2)^{1/2} P_l^{m\,\prime} = \frac{1}{2}P_l^{m+1} - \frac{1}{2}(l+m)(l-m+1)P_l^{m-1}$ $\quad \frac{d}{dx}P_l^m$

$$x = \cos\theta \xrightarrow{m=1} \sin\theta \frac{d}{d(\cos\theta)} P_l^1 = \frac{1}{2}P_l^2 - \frac{1}{2}(l+1)l \underbrace{P_l^0}_{P_l} \quad ; \quad \frac{d}{d(\cos\theta)}P_l^1 = \frac{-1}{\sin\theta}\frac{dP_l^1}{d\theta}$$

بازگشتی : $P_l^{m+1} - \dfrac{2mx}{(1-x^2)^{1/2}} P_l^m + \left[l(l+1) - m(m-1)\right]P_l^{m-1} = 0$

$$m=1,\ x=\cos\theta \Longrightarrow P_l^2 - \frac{2\cos\theta}{\sin\theta}P_l^1 + l(l+1)\underbrace{P_l^0}_{P_l} = 0$$

از روابط بازگشتی ۱ و ۲ : $\dfrac{d}{d\theta}P_l^1 = -\frac{1}{2}\left\{2\cot\theta P_l^1 - l(l+1)P_l\right\} + \frac{1}{2}l(l+1)P_l$

$$\overset{*}{\Longrightarrow} B_r = \frac{1}{r}\cot\theta \sum_{l=1} \cancel{C_l \left(\frac{a}{r}\right)^{l+1} P_l^1(\cos\theta)} - \frac{1}{r}\sum_{l=1} \cancel{C_l\left(\frac{a}{r}\right)^{l+1} \cot\theta P_l^1(\cos\theta)}$$

$$+ \frac{1}{r}\sum_{l=1} C_l \left(\frac{a}{r}\right)^{l+1} l(l+1)P_l(\cos\theta) \longrightarrow B_r = \sum_{l=1} \underbrace{C_l}_{?} \frac{a^{l+1}}{r^{l+2}} l(l+1) P_l(\cos\theta) \quad *'$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA) = \dots$$

به همین ترتیب B_θ را نیز محاسبه کنید (* تمرین)

قانون بیوساوار : $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\lambda} \times \vec{r}}{r^2} \longrightarrow \vec{B} = \frac{\mu_0 I}{2} \frac{a^2}{z^3} \left(1 + \frac{a^2}{z^2}\right)^{-3/2} \hat{z}$

روی محور z که عمود بر صفحه حلقه است.

$a \ll z$ $\theta = 0$, $B_r = B_z$. قرار دهیم $r = z$ در رابطه $*'$

$$\longrightarrow \left(1 + \frac{a^2}{z^2}\right)^{-3/2} \simeq 1 - \frac{3}{2}\left(\frac{a^2}{z^2}\right) + \frac{15}{8}\left(\frac{a^2}{z^2}\right)^2 + \cdots$$

$$\Longrightarrow \text{قانون بیوساوار } B = \frac{\mu_0 I}{2} \left\{ \boxed{\frac{a^2}{z^3}} - \frac{3}{2} \frac{a^4}{z^5} + \frac{15}{8} \frac{a^6}{z^7} + \cdots \right\}$$

$$= \frac{\mu_0 I}{2} \frac{a^2}{z^3} \sum_{s=0}^{\infty} (-1)^s \frac{(2s+1)!!}{(2s)!!} \left(\frac{a}{z}\right)^{2s} ; \; z > a$$

$$*' \longrightarrow B_z = C_1 \boxed{\frac{a^2}{z^3}} \underbrace{P_1(1)}_{1} \times 2 + C_2\, 2(3) \frac{a^3}{z^4} \underbrace{P_2(1)}_{1} + \cdots$$

$$\Longrightarrow C_1 = \frac{\mu_0 I}{4} \; , \; C_2 = 0 \; , \; C_3 = \frac{-\mu_0 I}{16} \; , \; C_4 = 0 \; , \cdots$$

رابطه کلی : $C_{2n+1} = (-1)^n \frac{\mu_0 I}{2} \frac{(2n-1)!!}{(2n+2)!!}$

هماهنگ‌های کروی

$\nabla^2 \psi + k^2 f(r) \psi = 0 \longrightarrow$ هلمهولتز

شرودینگر

لاپلاس

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial\psi}{\partial\theta}\right]\right) + \frac{1}{\sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} + l(l+1)\psi = 0 \qquad ; \; \psi = \psi(\theta, \phi) = \Theta(\theta)\, \Phi(\phi)$$

$$\frac{\sin\theta}{\Theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = -\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = m^2 \quad : \text{II}$$

$$\longrightarrow \frac{d^2\Phi}{d\phi^2} + m^2 \Phi = 0 \longrightarrow \Phi = A e^{im\phi} + B e^{-im\phi}$$

$$\Phi(\phi) = \Phi(\phi + 2\pi) \longrightarrow e^{im\phi} = e^{im(\phi + 2\pi)} \Longrightarrow e^{im2\pi} = 1 \longrightarrow m : \text{integer}$$

$$\int_0^{2\pi} \Phi^*_m(\phi)\, \Phi_{m'}(\phi)\, d\phi = 1 \longrightarrow A^2 \int_0^{2\pi} e^{-im\phi} e^{im'\phi} d\phi = 2\pi \delta_{mm'} A^2 = 1$$

$$\Longrightarrow A = \frac{1}{\sqrt{2\pi}}$$

$\text{III} : \frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$: قرار دادیم $\longrightarrow \Theta = P_l^m(\cos\theta)$

$$\Longrightarrow \psi(\theta, \phi) = \left[A e^{im\phi} + B e^{-im\phi}\right] P_l^m(\cos\theta)$$

می‌توان نوشت : $\psi(\theta, \phi) = A e^{im\phi} P_l^m(\cos\theta) + A' e^{-im\phi} P_l^{-m}(\cos\theta)$

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \Longrightarrow A' = (-1)^m \frac{(l+m)!}{(l-m)!} B$$

$$\longrightarrow \text{توابع لژاندر وابسته نرمالیزه} = \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)$$

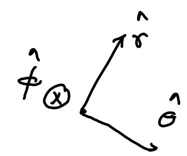
$$\boxed{\psi(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}} = Y_l^m(\theta,\phi) \quad ; \quad m = 0, \mp 1, \mp 2, \cdots$$

$$\Longrightarrow \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi\; Y_l^{m*}(\theta,\phi)\, Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'}$$

اندازه حرکت زاویه‌ای

$$\vec{L} = \vec{r} \times \vec{P} \longrightarrow \hat{\vec{L}} = \hat{\vec{R}} \times \hat{\vec{P}} \xrightarrow[\vec{P} = -i\hbar\vec{\nabla}]{\text{فضای مکان}} \vec{L} = -i\hbar\, \vec{R} \times \vec{\nabla} \quad , \vec{R} = r\hat{r}$$

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$$



$$\vec{L} = -i\hbar\left\{ \cancel{r}\frac{1}{\cancel{r}}\frac{\partial}{\partial\theta}\underbrace{\hat{r}\times\hat{\theta}}_{\hat{\phi}} + \cancel{r}\frac{1}{\cancel{r}\sin\theta}\frac{\partial}{\partial\phi}\underbrace{\hat{r}\times\hat{\phi}}_{-\hat{\theta}} \right\}$$

$$L_z = \vec{L}\cdot\hat{z}$$

$$= -i\hbar\left\{ \frac{\partial}{\partial\theta}\underbrace{\hat{\phi}\cdot\hat{z}}_{0} + \frac{-1}{\sin\theta}\frac{\partial}{\partial\phi}\overbrace{\hat{\theta}\cdot\hat{z}}^{-\sin\theta} \right\}$$

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{k} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi - \sin\theta\,\hat{k} \\ \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{cases}$$

$$\longrightarrow \boxed{L_z = -i\hbar\frac{\partial}{\partial\phi}}$$

به همین ترتیب L_x و L_y و L_+ و L_- را بیابید (* تمرین)

$$L_{\pm} = \frac{1}{2}(L_x \pm iL_y)$$

اثر L_z روی هماهنگ‌های کروی:

$$L_z Y_l^m(\theta,\phi) = \boxed{\text{ضریب}\; P_l^m(\cos\theta)} \underbrace{\frac{\partial}{\partial\phi} e^{im\phi}}_{im\, e^{im\phi}} (-i\hbar) = m\hbar\, Y_l^m(\theta,\phi)$$

$$L_z Y_l^m(\theta,\phi) = m\hbar\, Y_l^m(\theta,\phi)$$

ویژه مقدار — ویژه حالت — ویژه تابع

اثر L^2 روی هماهنگ‌های کروی:

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right\}$$

به عنوان تمرین حل کنید

$$L^2 Y_l^m(\theta,\phi) = -\hbar^2\,\text{ضریب}\left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) P_l^m(\cos\theta) e^{im\phi} + \frac{1}{\sin^2\theta} P_l^m \underbrace{\frac{\partial^2}{\partial\phi^2} e^{im\phi}}_{-m^2 e^{im\phi}} \right\}$$

$$= \hbar^2\, l(l+1)\, Y_l^m(\theta,\phi)$$

دیگر با لژاندر وابسته مقایسه شود:

اثر L_+ روی هماهنگ کروی:

$$L_+ = +\hbar e^{+i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right] \qquad Y_\ell^m = \text{ضریب}\; P_\ell^m(\cos\theta)\,e^{im\phi}$$

$$L_- Y_\ell^m(\theta,\phi) = +\hbar e^{i\phi}\left[\text{ضریب}\; e^{im\phi}\,\underbrace{\frac{d}{d\theta}P_\ell^m(\cos\theta)} + i\cot\theta \times \text{ضریب} \times P_\ell^m(\cos\theta)\,\underbrace{\frac{\partial}{\partial\phi}e^{im\phi}}_{im\,e^{im\phi}}\right]$$

مشتق زنجیره‌ای: $-\sin\theta\,\frac{d}{d(\cos\theta)}P_\ell^m(\cos\theta)$

$$= \hbar(-1)^{m+1}\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\;e^{i(m+1)\phi}\left[+\sin\theta\,\frac{dP_\ell^m}{\underbrace{d(\cos\theta)}_{?}} + m\cot\theta\,P_\ell^m\right]$$

داشتیم: $P_\ell^m(x) = (1-x^2)^{m/2}\,\frac{d^m P_\ell}{dx^m}$; $x = \cos\theta \longrightarrow P_\ell^m = \sin^m\theta\,\frac{d^m P_\ell}{dx^m}$

$$\Rightarrow \frac{d}{dx}P_\ell^m(x) = \frac{m}{\cancel{2}}\times(-\cancel{2}x)\left[1-x^2\right]^{m/2-1}\frac{d^m P_\ell}{dx^m} + (1-x^2)^{m/2}\,\frac{d^{m+1}P_\ell}{dx^{m+1}}$$

$$= -m\cos\theta\,\underbrace{(\sin^2\theta)^{m/2-1}}_{\sin^{-2}\theta\,\sin^m\theta}\,\frac{d^m P_\ell}{dx^m} + \sin^{m+1}\theta\,\frac{d^{m+1}P_\ell}{dx^{m+1}} \times \frac{1}{\sin\theta}$$

$P_\ell^m(\cos\theta)$ $\qquad$ $P_\ell^{m+1}(\cos\theta)$

$$\longrightarrow L_+ Y_\ell^m = (-1)^{m+1}\hbar\,e^{i(m+1)\phi}\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\left\{\cancel{-m\cos\theta/\sin\theta\,P_\ell^m} + P_\ell^{m+1} \cancel{+ m\cot\theta\,P_\ell^m}\right\}$$

$$Y_\ell^m = (-1)^m e^{im\phi}\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\;P_\ell^m$$

$$L_+ Y_\ell^m = \sqrt{\frac{\overbrace{\cancel{(\ell-m)!}}^{(\ell-m-1)!\,(\ell-m)}}{\cancel{(\ell+m)!}}}\;Y_\ell^{m+1}\sqrt{\frac{(\ell+m+1)!}{\cancel{(\ell-m-1)!}}} = \cancel{(\ell+m)!}\,(\ell+m+1)$$

$$\Rightarrow \boxed{L_+ Y_\ell^m(\theta,\phi) = \hbar\sqrt{(\ell-m)(\ell+m+1)}\;Y_\ell^{m+1}(\theta,\phi)}$$

*تمرین: اثر L_- را روی $Y_{\ell m}$ بیابید.