جلسه دوازدهم — معادله لژاندر وابسته ← توابع لژاندر وابسته — تابع مولد — روابط بازگشتی

تعامد $P_l^m$ ها

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**معادله لژاندر وابسته**

قضیه: اگر $z$ جواب معادله لژاندر باشد، آنگاه $(1-x^2)^{m/2}\dfrac{d^m z}{dx^m}$ جوابی از معادله لژاندر وابسته است.

$$(1-x^2)y''-2xy'+\left\{l(l+1)-\frac{m^2}{1-x^2}\right\}y=0$$

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}+l(l+1)z=0$$

از معادله فوق $m$ بار مشتق می‌گیریم:

$$\frac{d^m}{dx^m}\Big\{\overbrace{(1-x^2)}^{u}\overbrace{\frac{d^2z}{dx^2}}^{v}\Big\}-2\frac{d^m}{dx^m}\Big\{x\,\frac{dz}{dx}\Big\}+l(l+1)\frac{d^mz}{dx^m}=0$$

فرمول لایب‌نیتز:
$$\frac{d^m}{dx^m}(uv)=\sum_{r=0}^{m}\binom{m}{r}\frac{d^ru}{dx^r}\,\frac{d^{m-r}v}{dx^{m-r}}\quad;\quad\binom{m}{r}=\frac{m!}{r!\,(m-r)!}$$

$$\text{جمله اول}=\underbrace{\binom{m}{0}}_{1}(1-x^2)\underbrace{\frac{d^m}{dx^m}\frac{d^2z}{dx^2}}_{\frac{d^{m+2}z}{dx^{m+2}}}+\underbrace{\binom{m}{1}}_{\frac{m!}{(m-1)!}=m}\underbrace{\frac{d}{dx}(1-x^2)}_{-2x}\underbrace{\frac{d^{m-1}}{dx^{m-1}}\frac{d^2z}{dx^2}}_{\frac{d^{m+1}z}{dx^{m+1}}}+\underbrace{\binom{m}{2}}_{\frac{m!}{\cancel{2}!\,(m-2)!}=\frac12 m(m-1)}\frac{d^2}{dx^2}(1-x^2)\underbrace{\frac{d^{m-2}}{dx^{m-2}}\frac{d^2z}{dx^2}}_{d^mz/dx^m}$$

با $\frac{d^2}{dx^2}(1-x^2)=-\cancel{2}$.

$$\text{جمله دوم}=-2\left\{x\,\frac{d^{m+1}}{dx^{m+1}}z+m\times1\,\frac{d^m}{dx^m}z\right\}$$

$$(1-x^2)\frac{d^{m+2}z}{dx^{m+2}}-2x(m+1)\frac{d^{m+1}z}{dx^{m+1}}+\Big\{l(l+1)\underbrace{-m(m-1)-2m}_{-m^2+m-2m=-m^2-m=-m(m+1)}\Big\}\frac{d^mz}{dx^m}=0\qquad *$$

$$z_1\equiv\frac{d^mz}{dx^m}\quad,\quad z_2\equiv(1-x^2)^{m/2}z_1\;\Rightarrow\;z_2=(1-x^2)^{+m/2}\frac{d^mz}{dx^m}$$

$$(1-x^2)\frac{d^2}{dx^2}\Big\{(1-x^2)^{-m/2}z_2\Big\}-2x(m+1)\frac{d}{dx}\Big\{z_2(1-x^2)^{-m/2}\Big\}$$
$$+\Big\{l(l+1)-m(m+1)\Big\}(1-x^2)^{-m/2}z_2=0$$

$$\text{جمله دوم}=\frac{d}{dx}\Big\{z_2(1-x^2)^{-m/2}\Big\}=\frac{dz_2}{dx}(1-x^2)^{-m/2}+z_2\left(\cancel{-}\frac{m}{\cancel{2}}\right)\times(\cancel{-}\cancel{2}x)(1-x^2)^{-m/2-1}$$

$$\text{جمله اول}=\frac{d^2}{dx^2}\Big\{\quad''\quad\Big\}=\frac{d^2z_2}{dx^2}(1-x^2)^{-m/2}+\frac{dz_2}{dx}\left(\cancel{-}\frac{m}{\cancel{2}}\right)\times(\cancel{-}\cancel{2}x)(1-x^2)^{-m/2-1}$$
$$+\frac{dz_2}{dx}\,mx(1-x^2)^{-m/2-1}+z_2\,m(1-x^2)^{-m/2-1}+z_2\,mx\left(\cancel{-}\frac{m}{2}\cancel{-}1\right)\times(\cancel{-}2x)(1-x^2)^{-m/2-2}$$

با حذف فاکتور $(1-x^2)^{-m/2}$ در همه عبارت داریم:

$$(1-x^2)\frac{d^2 z_2}{dx^2} + \{\underbrace{2mx - 2(m+1)x}_{-2x}\}\frac{dz_2}{dx} + \{\cancel{m} + \underbrace{\frac{(m+2)m}{1-x^2}x^2 - \frac{2(m+1)mx^2}{1-x^2}}_{\frac{mx^2}{1-x^2}\{m+\cancel{2}-2m-\cancel{2}\}} + \underbrace{l(l+1)-m(m+1)}_{-m^2-\cancel{m}}\} z_2 = 0$$

$$\to (1-x^2)\frac{d^2 z_2}{dx^2} - 2x\frac{dz_2}{dx} + \{l(l+1)\underbrace{-\frac{m^2}{1-x^2}x^2 - m^2}_{-m^2\{\frac{\cancel{x^2}+1-\cancel{x^2}}{1-x^2}\}}\} z_2 = 0$$

$$\to \boxed{(1-x^2)\frac{d^2 z_2}{dx^2} - 2x\frac{dz_2}{dx^2} + \{l(l+1) - \frac{m^2}{1-x^2}\} z_2 = 0}$$

$$z_2 = (1-x^2)^{+m/2}\frac{d^m z}{dx^m} = P_l^m(x) \xrightarrow{z=P_l(x)} \boxed{P_l^m(x) = (1-x^2)^{m/2}\frac{d^m P_l}{dx^m}}$$

چندجمله‌ای مرتبه $l$

$m \le l$

رودریگز : $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

$$\to P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \;;\; l+m \ge 0 \Rightarrow m \ge -l$$

* تمرین: رابطه مقابل را ثابت کنید.

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

تابع مولد:

$$g(x,t) = \frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2xt+t^2)^{m+1/2}} = \sum_{r=0}^{\infty} P_{r+m}^m(x)\, t^r$$

خواص توابع لژاندر وابسته

1) $P_l^0(x) = P_l(x)$

2) if $m > l$ : $P_l^m(x) = 0$

3) $P_l^m(-x) = (-1)^{l+m} P_l^m(x)$

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \longrightarrow P_l(-x) = (-1)^l P_l(x)$$

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m P_l}{dx^m} \longrightarrow P_l^m(-x) = (1-x^2)^{m/2}\frac{d^m \overbrace{P_l(-x)}^{(-1)^l P_l(x)}}{\underbrace{d(-x)^m}_{(-1)^m dx^m}} = (-1)^{l+m} P_l^m(x)$$

4) $P_l^m(\mp 1) = 0$

روابط بازگشتی

1) $P_l^{m+1}(x) - \frac{2mx}{\sqrt{1-x^2}} P_l^m(x) + \{l(l+1) - m(m-1)\} P_l^{m-1}(x) = 0$

$$(1-x^2)\frac{d^{m+2}P_\ell(x)}{dx^{m+2}} - 2x(m+1)\frac{d^{m+1}P_\ell(x)}{dx^{m+1}} + \{\ell(\ell+1)-m(m+1)\}\frac{d^{m}P_\ell(x)}{dx^{m}} = 0$$ : رابطه *

طرفین را در $(1-x^2)^{m/2}$ ضرب می‌کنیم

$$\underbrace{(1-x^2)(1-x^2)^{m/2}}_{(1-x^2)^{\frac{m+2}{2}}}\frac{d^{m+2}}{dx^{m+2}}P_\ell - 2x(m+1)\underbrace{(1-x^2)^{m/2}\frac{d^{m+1}P_\ell}{dx^{m+1}}}_{(1-x^2)^{-1/2}P_\ell^{m+1}(x)} + \{\ell(\ell+1)-m(m+1)\}\underbrace{(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell}_{P_\ell^m(x)} = 0$$

$$P_\ell^{m+2} - 2x(m+1)(1-x^2)^{-1/2}P_\ell^{m+1} + \{\ell(\ell+1)-m(m+1)\}P_\ell^m = 0$$

$$m \to m-1 \Rightarrow P_\ell^{m+1} - \frac{2mx}{\sqrt{1-x^2}}P_\ell^m + \{\ell(\ell+1)-(m-1)m\}P_\ell^{m-1} = 0 \checkmark$$

2) $(2\ell+1)xP_\ell^m(x) = (\ell+m)P_{\ell-1}^m + (\ell-m+1)P_{\ell+1}^m(x)$

داریم: $(\ell+1)P_{\ell+1}(x) - (2\ell+1)xP_\ell(x) + \ell P_{\ell-1}(x) = 0$

m بار مشتق می‌گیریم:

$$(\ell+1)\frac{d^m}{dx^m}P_{\ell+1} - (2\ell+1)\underbrace{\frac{d^m}{dx^m}[xP_\ell]}_{x\frac{d^mP_\ell}{dx^m} + m\frac{d^{m-1}}{dx^{m-1}}P_\ell} + \ell\frac{d^m}{dx^m}P_{\ell-1} = 0 \qquad (1)$$

داریم: $P'_{\ell+1}(x) - P'_{\ell-1}(x) = (2\ell+1)P_\ell(x)$

m-1 بار مشتق ↓

$$\frac{d^mP_{\ell+1}}{dx^m} - \frac{d^mP_{\ell-1}}{dx^m} = (2\ell+1)\frac{d^{m-1}}{dx^{m-1}}P_\ell \qquad (2)$$

رابطه ۲ را در ۱ جایگذاری می‌کنیم:

$$\xrightarrow{(1-x^2)^{m/2}} (\ell+1)P_{\ell+1}^m - (2\ell+1)xP_\ell^m - m\{P_{\ell+1}^m - P_{\ell-1}^m\} + \ell P_{\ell-1}^m = 0$$

$$(2\ell+1)xP_\ell^m = P_{\ell+1}^m(\ell-m+1) + P_{\ell-1}^m(\ell+m) \checkmark$$

3) $\sqrt{1-x^2}\,P_\ell^m(x) = \frac{1}{2\ell+1}\{P_{\ell+1}^{m+1}(x) - P_{\ell-1}^{m+1}(x)\}$

$$(2)\xrightarrow{(1-x^2)^{m/2}} \underbrace{(1-x^2)^{m/2}\frac{d^mP_{\ell+1}}{dx^m}}_{P_{\ell+1}^m} - \underbrace{(1-x^2)^{m/2}\frac{d^mP_{\ell-1}}{dx^m}}_{P_{\ell-1}^m} = (2\ell+1)\underbrace{(1-x^2)^{m/2}}_{(1-x^2)^{1/2}(1-x^2)^{\frac{m-1}{2}}}\frac{d^{m-1}P_\ell}{dx^{m-1}}$$

$P_\ell^{m-1}$

$$P_{l+1}^{m} - P_{l-1}^{m} = \sqrt{1-x^2}\,(2l+1)\,P_l^{m-1} \quad \text{(A)}$$

$$m \to m+1 \;\longrightarrow\; \sqrt{1-x^2}\,P_l^{m} = \frac{1}{2l+1}\left\{ P_{l+1}^{m+1} - P_{l-1}^{m+1} \right\} \quad \checkmark$$

$$4)\; \sqrt{1-x^2}\,P_l^{m}(x) = \frac{1}{2l+1}\left\{ (l+m)(l+m-1)\,P_{l-1}^{m-1}(x) - (l-m+1)(l-m+2)\,P_{l+1}^{m-1}(x) \right\}$$

$$2)\; (2l+1)\,x P_l^{m}(x) = (l+m)P_{l-1}^{m} + (l-m+1)P_{l+1}^{m}(x)$$

$$1)\; P_l^{m+1}(x) - \frac{mx}{\sqrt{1-x^2}}\,P_l^{m}(x) + \left\{ l(l+1) - m(m-1) \right\} P_l^{m-1}(x) = 0$$

$$\Rightarrow\; P_l^{m+1} - \frac{2m}{\sqrt{1-x^2}} \times \frac{1}{2l+1}\left\{ (l+m)\,P_{l-1}^{m} + (l-m+1)\,P_{l+1}^{m} \right\} + \left\{ l(l+1) - m(m-1) \right\} P_l^{m-1} = 0$$

در رابطه A جایگذاری می‌کنیم

$$\Rightarrow \sqrt{1-x^2}\,P_l^{m+1} - \frac{2m}{2l+1}\left\{ (l+m)\,\boxed{P_{l-1}^{m}} + (l-m+1)\,\boxed{P_{l+1}^{m}} \right\} + \left\{ l(l+1) - m(m-1) \right\} \frac{1}{2l+1}\left[ \boxed{P_{l+1}^{m}} - \boxed{P_{l-1}^{m}} \right] = 0$$

$$\sqrt{1-x^2}\,P_l^{m+1} = \frac{1}{2l+1}\Big[ P_{l-1}^{m}\underbrace{\left\{ 2m(l+m) + l(l+1) - m(m-1) \right\}}_{\substack{2ml + 2m^2 + l^2 + l - m^2 + m \\ = (l+m)(l+m+1)}} + P_{l+1}^{m}\underbrace{\left\{ 2m(l-m+1) - l(l+1) + m(m-1) \right\}}_{\substack{2ml - 2m^2 + 2m - l^2 - l + m^2 - m \\ = -(l-m)(l-m+1)}}$$

$$m \to m-1 :\; \sqrt{1-x^2}\,P_l^{m} = \frac{1}{2l+1}\left[ P_{l-1}^{m-1}(l+m-1)(l+m) - P_{l+1}^{m-1}(l-m+1)(l-m+2) \right] \quad \checkmark$$

تعامد $P_l^m$ ها

$$\int_{-1}^{+1} P_l^{m}(x)\,P_{l'}^{m}(x)\,dx = \frac{2}{2l+1}\,\frac{(l+m)!}{(l-m)!}\,\delta_{ll'}$$

$$\int_{-1}^{+} \frac{1}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \,\frac{1}{2^{l'} l'!}(1-x^2)^{m/2}\frac{d^{l'+m}}{dx^{l'+m}}(x^2-1)^{l'}\,dx$$

$$= \frac{(-1)^m}{2^{l+l'}\,l!\,l'!}\int_{-1}^{+1} \underbrace{(x^2-1)^m\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l}_{u}\;\underbrace{\frac{d^{l'+m}}{dx^{l'+m}}(x^2-1)^{l'}}_{dv}$$

فرض کنید $l' < l$

انتگرال جزء به جزء به اندازه $l'+m$ بار می‌گیریم. قسمت‌های مرزی که شامل جمله $(x^2-1)$ هستند صفر می‌شوند

$$= \frac{(-1)^m}{2^{l+l'}\,l!\,l'!}(-1)^{l'+m}\int_{-1}^{+1} \frac{d^{l'+m}}{dx^{l'+m}}\left[ (x^2-1)^m \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \right](x^2-1)^{l'}\,dx \quad \text{(B)}$$

$$\text{انتگرال جزء به جزء بار اول} = \left.(x^2-1)^m \frac{d^{l+m}}{dx^{l+m}}(x^2-1)\,\frac{d^{l'+m-1}}{dx^{l'+m-1}}(x^2-1)^{l'}\right|_{-1}^{+1}$$

$$-\int_{-1}^{+1}\frac{d^{l'+m-1}}{dx^{l'+m-1}}(x^2-1)^{l'}\ \frac{d}{dx}\left[(x^2-1)^m\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l\right]$$

مشتق رابطه ③ را از فرمول لایبنیز باز کنید :

$$(x^2-1)^{l'}\frac{d^{l'+m}}{dx^{l'+m}}\left[\underbrace{(x^2-1)^m}_{v}\ \underbrace{\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l}_{u}\right]=$$

$$\frac{d^m}{dx^m}(uv)=\sum_{r=0}^{m}\binom{m}{r}\frac{d^r u}{dx^r}\frac{d^{m-r}v}{dx^{m-r}}$$

$$(x^2-1)^{l'}\sum_{r=0}^{l'+m}\underbrace{\binom{l'+m}{r}}_{\frac{(l'+m)!}{(l'+m-r)!\,r!}}\ \underbrace{\frac{d^r}{dx^r}\frac{d^{l+m}}{dx^{l+m}}}_{\frac{d^{l+m+r}}{dx^{l+m+r}}}(x^2-1)^l\ \underbrace{\frac{d^{l'+m-r}}{dx^{l'+m-r}}(x^2-1)^m}_{\downarrow}$$

$$l'+m-r\leqslant 2m \Longrightarrow r\geqslant l'-m$$

$$\frac{d^{l+m+r}}{dx^{l+m+r}}(x^2-1)^l \rightsquigarrow l+m+r\leqslant 2l \Rightarrow r\leqslant l-m \longrightarrow l'-m\leqslant r\leqslant l-m$$

با توجه به فرض که $l\leqslant l'$ $\Longleftarrow$ $l=l'$ , $r=l-m$

$$\int_{-1}^{+1}[P_l^m(x)]^2dx=\frac{(-1)^{l+2m}}{2^{2l}[l!]^2}\ \frac{(l+m)!}{(l-m)!\,\underbrace{(\cancel{l}+m-(\cancel{l}-m))}_{(2m)}!}\int_{-1}^{+1}(x^2-1)^l\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

$$\times\ \frac{d^{\cancel{l}+m-(\cancel{l}-m)}}{dx^{\cancel{l}+m-(\cancel{l}-m)}}(x^2-1)^m$$

$$=\frac{(-1)^{l+2m}}{2^{2l}[l!]^2}\ \frac{(l+m)!}{(l-m)!\,\cancel{(2m)!}}\int_{-1}^{+1}(x^2-1)^l\ \underbrace{\frac{d^{2m}}{dx^{2m}}(x^2-1)^m}_{\cancel{(2m)!}}\ \underbrace{\frac{d^{2l}}{dx^{2l}}(x^2-1)^l}_{(2l)!}\,dx$$

$$\int_{-1}^{+1}(x^2-1)^l dx=(-1)^l\int_{-1}^{+1}(1-x^2)^l dx=(-1)^l\int_0^1(1-t)^l t^{-1/2}dt=\beta(1/2,\,l+1)(-1)^l$$

داشتیم

$$x^2=t \longrightarrow dt=2x\,dx$$

$$=(-1)^l\frac{\Gamma(1/2)\Gamma(l+1)}{\Gamma(l+3/2)}=(-1)^l\frac{\sqrt{\pi}\ l!}{(l+1/2)!}=(-1)^l\frac{l!\,2^{l+2}(l+1)!}{(2l+2)!}$$

$$\Gamma(l+1/2)=\frac{(2l)!}{2^{2l}\,l!}\sqrt{\pi}\longrightarrow \Gamma(l+3/2)=(l+1/2)!=\frac{(2l+2)!\sqrt{\pi}}{2^{2l+2}(l+1)!}$$

$$\longrightarrow \int_{-1}^{1}P_l^m(x)\,P_{l'}^m(x)\,dx=\frac{2}{2l+1}\ \frac{(l+m)!}{(l-m)!}\ \delta_{ll'}$$