جلسه دهم : تابع مولد چندجمله‌ای های لژاندر   تابع پتانسیل بار نقطه‌ای   تابع پتانسیل چندقطبی   یک سری روابط

تعامد توابع لژاندر   روابط بازگشتی

یادآوری : $P_l(x) = \dfrac{1}{2^l} \displaystyle\sum_{r=0}^{[l/2]} (-1)^r \dfrac{(2l-2r)!}{r!\,(l-r)!} \dfrac{x^{l-2r}}{(l-2r)!}$

$$[l/2] = \begin{cases} \frac{1}{2} l & l : \text{even} \\ \frac{l-1}{2} & l : \text{odd} \end{cases} \qquad (1-x^2)y'' - 2xy' + l(l+1)y = 0$$

تابع مولد

قضیه : اگر $|t|<1$ و $|x| \leqslant 1$ باشد، داریم :

اثبات :

$$\frac{1}{\sqrt{t^2 - 2tx + 1}} = g(x,t) = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

یادآوری : $(a+b)^N = \displaystyle\sum_{m=0}^{N} \binom{N}{m} a^m b^{N-m}$; $\dbinom{N}{m} = \dfrac{N!}{m!\,(N-m)!}$

$$(t^2 - 2tx + 1)^{-1/2} = \left[1 - t(2x-t)\right]^{-1/2} = 1 + (-\tfrac{1}{2})\left[-t(2x-t)\right] + \frac{1}{2!}(-\tfrac{1}{2})(-\tfrac{3}{2})\left[-t(2x-t)\right]^2$$

$$+ \cdots + \frac{1}{r!}(-\tfrac{1}{2})(-\tfrac{3}{2})\cdots\left(-\frac{2r-1}{2}\right)\left[-t(2x-t)\right]^r + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{2^r} \boxed{1\cdot3\cdot5\cdots(2r-1)} \cancel{(-1)^r} \cancel{(-1)^r} t^r (2x-t)^r \times \frac{\boxed{2\cdot4\cdot6\cdots2r}}{2\cdot4\cdot6\cdots2r} \rightarrow (2r)!$$

$$2\cdot4\cdot6\cdots2r = 2^r(1\cdot2\cdot3\cdots r)$$

$$= \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \frac{(2r)!}{2^{2r}} t^r \underbrace{(2x-t)^r} \longrightarrow \sum_{p=0}^{r} \binom{r}{p} (2x)^{r-p} (-t)^p$$

$$(t^2 - 2tx + 1)^{-1/2} = \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r}(r!)^2} \sum_{p=0}^{r} \binom{r}{p} (-1)^p t^{r+p} (2x)^{r-p} = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

$$r+p = l \Longrightarrow p = l-r \longrightarrow \begin{cases} p=0 \rightarrow r=l \\ p=r \rightarrow r=l/2 \end{cases} \qquad l/2 \leqslant r \leqslant l$$

اگر l عدد فرد باشد : $\frac{l+1}{2} \leqslant r \leqslant l$

$$\text{ضریب } t^l = \sum_{r=l/2 \text{ یا } \frac{l+1}{2}}^{l} \frac{(2r)!}{2^{2r}(r!)^2} \binom{r}{l-r} (-1)^{l-r} (2x)^{2r-l}$$

تغییر متغیر : $k = l-r \rightarrow r=l \rightarrow k=0$

$$k = [l/2] \leftarrow k = l/2 \text{ یا } k = \frac{l-1}{2} \leftarrow r = l/2 \text{ یا } r = \frac{l+1}{2}$$

$$\text{ضریب } t^l = \sum_{k=0}^{[l/2]} \frac{(2l-2k)!}{2^l\,\cancel{2^{2l-2k}}\,\left[(l-k)!\right]^{\cancel{2}}} \frac{\cancel{(l-k)!}}{(l-2k)!\,k!} (-1)^k\, 2^{\cancel{l-2k}} x^{l-2k} = P_l(x)$$

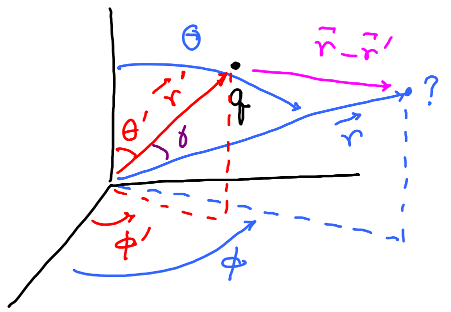
بررسی شرط $|t|<1$ و $|x|\leqslant 1$:

$$\left[1-t(2x-t)\right]^{-1/2} \rightsquigarrow |t(2x-t)|<1$$

$$\longrightarrow |t|\,|2x-t|<1 \Longrightarrow |t|\left\{|2x|-|t|\right\}<1 \Longrightarrow 2|x||t|-|t|^2<1$$

$$\longrightarrow |x|<\frac{1+|t|^2}{2|t|} \xrightarrow{|x|\leqslant 1} 1+|t|^2<2|t| \qquad \times$$

پتانسیل بار نقطه‌ای الکتریکی



$$\varphi(r)=\frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec r-\vec r'|}$$

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

* اثبات به عهده دانشجو

الف - $r>r' \rightarrow t(r'/r)<1$

$$\frac{1}{|\vec r-\vec r'|}=(r^2+r'^2-2\underbrace{\vec r\cdot\vec r'}_{rr'\cos\gamma})^{-1/2}=r^{-1}\left[1-2\frac{r'}{r}\cos\gamma+\left(\frac{r'}{r}\right)^2\right]^{-1/2}=\sum_{l=0}^{\infty}r^{-1}P_l(\cos\gamma)\left(\frac{r'}{r}\right)^l$$

$$\frac{1}{\sqrt{t^2-2tx+1}}=\sum P_l(x)\,t^l \qquad |t|<1$$

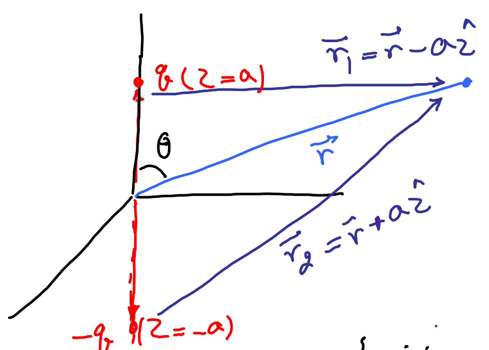
$$\Longrightarrow \varphi(r)=\frac{q}{4\pi\epsilon_0}\frac{1}{r}\sum_{l=0}^{\infty}P_l(\cos\gamma)\left(r'/r\right)^l$$

ب - $r'>r \rightarrow t=r/r'<1$

$$\frac{1}{|\vec r-\vec r'|}=\sum_{l=0}^{\infty}r'^{-1}P_l(\cos\gamma)\left(r/r'\right)^l$$

$$\longrightarrow \boxed{\varphi(r)=\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}P_l(\cos\gamma)\frac{r_<^l}{r_>^{l+1}}} \xrightarrow[\text{محور z باشد}]{\text{بار الکتریکی روی}} \cos\gamma=\cos\theta$$

پتانسیل چندقطبی الکتریکی



$$\varphi(r)=\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec r_1|}-\frac{1}{|\vec r_2|}\right\}$$

$$\frac{1}{|\vec r_1|}=\frac{1}{|\vec r-a\hat z|}=(r^2+a^2-2a\underbrace{\vec r\cdot\hat z}_{r\cos\theta})^{-1/2}$$

فرض شده : $a<r \Longrightarrow a/r=t<1$

$$\frac{1}{|\vec r_1|}=r^{-1}\left[(a/r)^2-2\,a/r\,\cos\theta+1\right]^{-1/2}=r^{-1}\sum_{l=0}^{\infty}P_l(\cos\theta)\,(a/r)^l$$

$$\varphi(r)=\frac{q}{4\pi\epsilon_0 r}\left\{\sum_{l=0}^{\infty}P_l(\cos\theta)\,(a/r)^l-\sum_{l=0}^{\infty}P_l(\cos\theta)\,(-1)^l\,(a/r)^l\right\}$$

$$\varphi(r) = \frac{2q}{4\pi\epsilon_0 r}\left[P_1(\cos\theta)\frac{a}{r} + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + \cdots\right]$$

اثر $O(a/r)^3 \longrightarrow 0$: $\varphi(r) = \dfrac{2qa}{4\pi\epsilon_0}\dfrac{P_1(\cos\theta)}{r^2}$: پتانسیل دوقطبی

یک سری روابط

1) $P_\ell(-1) = (-1)^\ell$

$$\frac{1}{\sqrt{t^2-2tx+1}} = \sum_{\ell=0}^{\infty} P_\ell(x)\, t^\ell \xrightarrow{x=-1} g(x=-1,t) = \frac{1}{\sqrt{t^2+2t+1}} = \frac{1}{1+t}$$

یادآوری: $1+t+t^2+\cdots \overset{|t|<1}{=} \dfrac{1}{1-t} \Longrightarrow \dfrac{1}{1+t} = \displaystyle\sum_{\ell=0}^{\infty}(-1)^\ell t^\ell = 1-t+t^2+\cdots$

$$g(-1,t) = \sum (-1)^\ell t^\ell = \sum P_\ell(-1)\, t^\ell \Longrightarrow P_\ell(-1) = (-1)^\ell \checkmark$$

۲) $P_\ell'(1) = \frac{1}{2}\ell(\ell+1)$

$$(1-x^2)P_\ell''(x) - 2x P_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0 \xrightarrow{x=1} -2P_\ell'(1) + \ell(\ell+1)\underbrace{P_\ell(1)}_{1} = 0$$

$$\Longrightarrow P_\ell'(1) = \frac{1}{2}\ell(\ell+1) \checkmark$$

$\downarrow x=-1$

۳) $P_\ell'(-1) = (-1)^{\ell-1}\frac{1}{2}\ell(\ell+1)$

$$+2P_\ell'(-1) + \ell(\ell+1)\underbrace{P_\ell(-1)}_{(-1)^\ell} = 0$$

۴) $P_{2\ell}(0) = (-1)^\ell \dfrac{(2\ell)!}{2^{2\ell}(\ell!)^2}$

۵) $P_{2\ell+1}(0) = 0$

$$\frac{1}{\sqrt{t^2-2tx+1}} = \sum_{\ell=0}^{\infty} P_\ell(x)\, t^\ell$$

$$x=0 : \quad g(x=0,t) = \frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2} \overset{\text{بسط دوجمله‌ای}}{=} 1 + \left(-\tfrac{1}{2}\right)t^2 + \frac{1}{2!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)(t^2)^2 + \cdots$$

$$\Longrightarrow (1+t^2)^{-1/2} = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(-1)^\ell\, \frac{\boxed{1.3.5\cdots(2\ell-1)}}{2^\ell}(t^2)^\ell \times \frac{\boxed{2.4.6\cdots 2\ell}}{2.4.6\cdots 2\ell}$$

$(2\ell)!$

$= 2^\ell\, \ell!$

$$= \underbrace{\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2^{2\ell}}\frac{(2\ell)!}{[\ell!]^2}}\, t^{2\ell} \overset{\text{باید}}{=} \sum_{\ell=0}^{\infty} P_\ell(0)\, t^\ell$$

$P_{2\ell}(0)$, $P_{2\ell+1}(0) = 0$

$$\int_{-1}^{+1} P_l(x)\,P_{l'}(x)\,dx = \frac{2}{2l+1}\,\delta_{ll'}$$

تعامد توابع لژاندر : اگر l و l' دو عدد صحیح غیر منفی دلخواه باشند

فرض کنید $l' \neq l$ است:

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \longrightarrow \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + l(l+1)y = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] + l(l+1)P_l = 0 \xrightarrow{\times P_{l'}}$$

دو معادله را کم کرده و از $[-1, +1]$ انتگرال می‌گیریم

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_{l'}}{dx}\right] + l'(l'+1)P_{l'} = 0 \xrightarrow{\times P_l}$$

$$\int_{-1}^{+1}\left[\boxed{P_{l'}\frac{d}{dx}\left\{(1-x^2)\frac{dP_l}{dx}\right\}} - P_l\frac{d}{dx}\left\{(1-x^2)\frac{dP_{l'}}{dx}\right\}\right]dx +$$

$$\left[l(l+1) - l'(l'+1)\right]\int_{-1}^{+1} P_l\,P_{l'}\,dx = 0 \qquad *$$

$$\frac{d}{dx}\left[P_{l'}\underbrace{(1-x^2)\frac{dP_l}{dx}}\right] = \frac{dP_{l'}}{dx}(1-x^2)\frac{dP_l}{dx} + \boxed{P_{l'}\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right]}$$

جمله اول* $$= \int_{-1}^{+1}\frac{d}{dx}\left[P_{l'}\cancelto{0}{(1-x^2)}\frac{dP_l}{dx}\right]dx - \int_{-1}^{+1}\frac{dP_{l'}}{dx}\cancel{(1-x^2)}\frac{dP_l}{dx}\,dx$$

$$-\underbrace{\int_{-1}^{+1}\frac{d}{dx}\left[P_l\cancelto{0}{(1-x^2)}\frac{dP_{l'}}{dx}\right]dx}_{P_l \times (1-x^2)\frac{dP_{l'}}{dx}\Big|_{-1}^{+1} = 0} + \int_{-1}^{+1}\frac{dP_l}{dx}\cancel{(1-x^2)}\frac{dP_{l'}}{dx}\,dx = 0$$

$l \neq l'$

$$\longrightarrow *:\ (\underbrace{l^2 - l - l'^2 - l'}_{l\neq l' \to\ \neq 0})\int_{-1}^{+1} P_l(x)\,P_{l'}(x)\,dx = 0 \longrightarrow \boxed{\int_{-1}^{+1} P_l(x)\,P_{l'}(x)\,dx = 0}$$

فرض کنیم $l = l'$:

$$\int_{-1}^{+1} P_l^2(x)\,dx = \frac{2}{2l+1}$$

$$\frac{1}{\sqrt{t^2 - 2tx + 1}} = \sum_{l=0}^{\infty} P_l(x)\,t^l \longrightarrow \frac{1}{t^2 - 2tx + 1} = \sum_{l=0} P_l\,t^l \sum_{l'=0}^{\infty} P_{l'}\,t^{l'}$$

$$= \sum_{l,l'=0} P_l(x)\,P_{l'}(x)\,t^{l+l'}$$

$$\longrightarrow \int_{-1}^{+1}\frac{dx}{t^2 - 2tx + 1} = \sum_{l,l'} t^{l+l'}\int_{-1}^{+1} P_l(x)\,P_{l'}(x)\,dx \quad **$$

$$\xrightarrow{l=l'} \sum_{l} t^{2l}\int_{-1}^{+1} P_l^2(x)\,dx$$

$$\int \frac{d}{a+bx} = \frac{1}{b}\ln(a+bx) + C, \quad a = t^2+1,\ b = -2t$$

$$L.H.S ** = \frac{1}{-2t}\ln(t^2-2tx+1)\Big|_{-1}^{+1} = -\frac{1}{2t}\Big[\ln(\underbrace{t^2-2t+1}_{(1-t)^2}) - \ln(\underbrace{t^2+2t+1}_{(t+1)^2})\Big]$$

$$= -\tfrac{1}{t}\Big[\ln(1-t) - \ln(t+1)\Big] = \tfrac{1}{t}\Big[t - \tfrac{t^2}{2} + \tfrac{t^3}{3} - \cdots - (-t - \tfrac{t^2}{2} - \tfrac{t^3}{3} - \cdots)\Big]$$

$$= \tfrac{1}{t}\Big[2t + \tfrac{2t^3}{3} + \cdots\Big] = 2\left(1 + \tfrac{t^2}{3} + \tfrac{t^4}{5} + \cdots\right)$$

$$\Rightarrow L.H.S ** = 2\sum_{l=0}^{\infty}\frac{t^{2l}}{(2l+1)} = R.H.S ** \longrightarrow \int_{-1}^{+1} P_l^2(x)\,dx = \frac{2}{2l+1} \quad ✓$$

روابط بازگشتی چندجمله‌ای‌های لژاندر

$$1)\ (l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0$$

$$g(x,t) = \frac{1}{\sqrt{t^2-2tx+1}} = \sum P_l t^l \Rightarrow \frac{\partial g}{\partial t} = \frac{x-t}{\underbrace{(1-2xt+t^2)^{3/2}}_{(1-2xt+t^2)^{1/2}\,(1-2xt+t^2)}} = \sum_{l=0} P_l\, l\, t^{l-1}$$

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{l=0} P_l\, l\, t^{l-1}$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

$$\longrightarrow (1-2xt+t^2)\sum_{l=0} P_l\, l\, t^{l-1} + (t-x)\sum_{l=0} P_l(x)\, t^l = 0$$

$$\longrightarrow \sum_{l=0} P_l\, l\, t^{l+1} - 2x\sum_{l=0} P_l\, l\, t^l - x\sum_{l=0} P_l\, t^l + \sum_{l=0} P_l\, l\, t^{l-1} + \sum_{l=0} P_l\, t^{l+1} = 0$$

$$l+1 = l' \Rightarrow l = l'-1 \longrightarrow l=0 \equiv l'=1$$

هم‌توانها را تبدیل به t^l می‌کنیم:

$$\text{جمله اول} = \sum_{l'=1} P_{l'-1}\,(l'-1)\, t^{l'}$$

جمع صفر در $l=-1$

$$\sum_{l=1} P_{l-1}(l-1)\,t^l - 2x\sum_{l=0} P_l\, l\, t^l - x\sum_{l=0} P_l\, t^l + \sum_{\cancel{l=-1}\ l=0} P_{l+1}(l+1)\, t^l + \sum_{l=1} P_{l-1}\, t^l = 0$$

$$0 = -2xP_0 \times 0 - x\underbrace{P_0}_{1} + P_1^{=x} = \quad t^0 \text{ ضریب}$$

$$\text{برای } l \geq 1 \longrightarrow (l\cancel{-1})P_{l-1} - (2l+1)xP_l + (l+1)P_{l+1} + \cancel{P_{l-1}} = 0$$

$$\Rightarrow lP_{l-1} - (2l+1)xP_l + (l+1)P_{l+1} = 0 \quad ✓$$