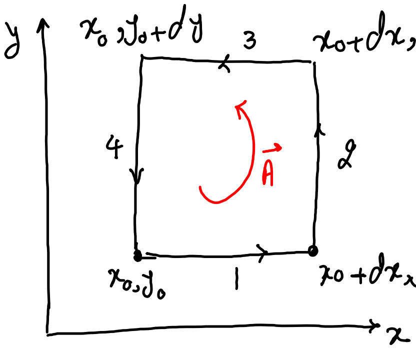
جلسه چهارم، تعبیر فیزیکی کرل   کاربردهای $\vec{\nabla}$ ← معادلات موج E.M.

کرل $\vec{\nabla}\times$ curl



$$\text{چرخش}\Big|_{1234} = \int_{x_0}^{x_0+dx} A_x\,dx + \int_{y_0}^{y_0+dy} A_y\,dy + \int_{x_0+dx}^{x_0} A_x\,dx + \int_{y_0+dy}^{y_0} A_y\,dy = \oint \vec{A}\cdot d\vec{l}$$

فرض: $\vec{A}$ تابعی پیوسته از x و y است ← قضیه مقدار میانگین

$$\int_a^b f(x)\,dx = f(\xi)\,(a-b) \qquad \xi\in[a,b]$$

$$\oint \vec{A}\cdot d\vec{l} = A_x(x_0,y_0)\,dx + \underbrace{A_y(\overset{\epsilon}{x_0+dx},y_0)\,dy}_{\mathrm{I}} - \underbrace{A_x(x_0,\overset{\epsilon}{y_0+dy})\,dx}_{\mathrm{II}} - A_y(x_0,y_0)\,dy$$

بسط تیلور: $f(x+\epsilon) = f(x) + \epsilon\, \dfrac{df}{dx} + \dfrac{1}{2!}\dfrac{d^2f}{dx^2}\epsilon^2 + \cdots$

$$\mathrm{I} = \left[A_y(x_0,y_0) + \frac{\partial A_y}{\partial x}\Big|_{x_0,y_0} dx + \cdots\right] dy$$

$$\mathrm{II} = \left[A_x(x_0,y_0) + \frac{\partial A_x}{\partial y}\Big|_{x_0,y_0} dy + \cdots\right] dx$$

چرخش:

$$\Rightarrow \oint \vec{A}\cdot d\vec{l} = \cancel{A_x(x_0,y_0)\,dx} + \cancel{A_y(x_0,y_0)\,dy} + \frac{\partial A_y}{\partial x}\Big|_{x_0,y_0} dx\,dy - \cancel{A_x(x_0,y_0)\,dx} - \frac{\partial A_x}{\partial y}\Big|_{x_0,y_0} dy\,dx - \cancel{A_y(x_0,y_0)\,dy} = \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{(\vec{\nabla}\times\vec{A})_z}\Bigg|_{x_0,y_0} dx\,dy$$

$\Rightarrow$ چرخش در واحد سطح $= (\vec{\nabla}\times\vec{A})_z$

عمود بر محور z

مثال در فیزیک: الکتروستاتیک - بدون چرخش: $V \propto \dfrac{\vec{r}}{r^3} = \dfrac{\hat{r}}{r^2}$

irrotational میدانهای غیرچرخشی ← $\vec{\nabla}\times\vec{V} = 0$

میدان چرخشی: $\vec{\nabla}\times\vec{A} = \vec{B}$

کاربردهای $\vec{\nabla}$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \underbrace{\vec{B}\times(\vec{\nabla}\times\vec{A})}_{III} + \underbrace{\vec{A}\times(\vec{\nabla}\times\vec{B})}_{IV}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\overset{A}{} \quad \overset{B}{} \quad \overset{C}{} \qquad \overset{B}{} \quad \overset{A\cdot C}{} \qquad \overset{C \; A\cdot B}{}$$

$$III = \vec{B}\times(\vec{\nabla}_A\times\vec{A}) = \vec{\nabla}_A(\vec{B}\cdot\vec{A}) - \vec{A}(\vec{B}\cdot\vec{\nabla}_A) = \vec{\nabla}_A(\vec{B}\cdot\vec{A}) - (\vec{B}\cdot\vec{\nabla}_A)\vec{A}$$

$$IV = \vec{A}\times(\vec{\nabla}_B\times\vec{B}) = \vec{\nabla}_B(\vec{A}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\vec{\nabla}_B) = \vec{\nabla}_B(\vec{A}\cdot\vec{B}) - (\vec{A}\cdot\vec{\nabla}_B)\vec{B}$$

$$III + IV = \underbrace{\vec{\nabla}_A(\vec{A}\cdot\vec{B}) + \vec{\nabla}_B(\vec{A}\cdot\vec{B})}_{\vec{\nabla}(\vec{A}\cdot\vec{B})} - (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B}$$

$$R.H.S = \vec{\nabla}(\vec{A}\cdot\vec{B}) = L.H.S$$

الف - دیورژانس گرادیان

$$\vec{\nabla}\cdot\vec{\nabla}\varphi = \nabla^2\varphi = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\frac{\partial\varphi}{\partial x}\hat{x} + \frac{\partial\varphi}{\partial y}\hat{y} + \frac{\partial\varphi}{\partial z}\hat{z}\right)$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \rightarrow \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$: لاپلاسین

دالامبری: $\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ ← معادله فیزیکی : $(\Box + m^2)\varphi = 0$ (معادله کلین-گوردون)

معادله فیزیکی : $\nabla^2\varphi = 0$ معادله لاپلاس ; $\nabla^2\varphi = \rho/\varepsilon_0$, ρ = چگالی بار

φ پتانسیل اسکالر است. ← معادله پواسون

مثال : $\vec{\nabla}\cdot\vec{\nabla}f(r) = ?$

میدانیم : $\vec{\nabla}f(r) = \frac{df}{dr}\hat{r}$ * ← جهت شعاعی

$\hat{r}$, $\hat{r}$, r

$$\vec{\nabla}\cdot\vec{\nabla}f(r) = \vec{\nabla}\cdot\left(\frac{df}{dr}\frac{\vec{r}}{r}\right)$$

$$\vec{\nabla}\cdot(f\vec{v}) = \vec{\nabla}f\cdot\vec{v} + f\,\vec{\nabla}\cdot\vec{v}$$

$$= \left\{\vec{\nabla}\left(\frac{df}{dr}\frac{1}{r}\right)\right\}\cdot\vec{r} + \frac{1}{r}\frac{df}{dr}\underbrace{(\vec{\nabla}\cdot\vec{r})}_{3}$$

$$\vec{r} = (x\hat{x} + y\hat{y} + z\hat{z}), \quad \hat{r} = \vec{r}/r$$

جمله اول $= \frac{d}{dr}\left(\frac{df}{dr}\frac{1}{r}\right)\hat{r}\cdot\vec{r} = \left\{\frac{1}{r}\frac{d^2f}{dr^2} + \frac{df}{dr}\overbrace{\frac{d}{dr}(1/r)}^{-1/r^2}\right\}\underbrace{\hat{r}\cdot\vec{r}}_{r}$ *

$$= \frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr}$$

$\vec{\nabla}\cdot\vec{r} = 3$

$\vec{\nabla}\cdot\hat{r}r = 3$

$\Rightarrow \vec{\nabla}\cdot\hat{r} = 3/r$

$$\rightarrow \vec{\nabla}\cdot\vec{\nabla} f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

حالت خاص :
$$\nabla^2 r^n = n(n-1)r^{n-2} + \frac{2}{r} n r^{n-1} = r^{n-2}\left(n(n-1)+2n\right) = (n^2+n)r^{n-2}$$

$$f(r) = r^n \longrightarrow \frac{df}{dr} = n r^{n-1} \quad , \quad \frac{d^2 f}{dr^2} = n(n-1) r^{n-2}$$

اگر

$$n=-1 : \quad f(r) = \frac{1}{r} \Rightarrow \nabla^2 f(r) = \nabla^2\left(\frac{1}{r}\right) = r^{-3}(1-1) = 0 \Rightarrow \boxed{\nabla^2\left(\frac{1}{r}\right) = 0}$$

ب - کرل گرادیان

$$\vec{\nabla}\times\vec{\nabla}\varphi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = \hat{x}\underbrace{\left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)}_{0} + \hat{y}\underbrace{(\quad)}_{0} + \hat{z}\underbrace{(\quad)}_{0} = 0$$

کرل هر گرادیانی برابر صفر است.

ج - دیورژانس کرل

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = 0$$

دیورژانس هر کرل برابر صفر است.

د - کرل کرل

$A \quad B \quad C \qquad B(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}_A(\vec{\nabla}\cdot\vec{A}) - \vec{A}(\vec{\nabla}\cdot\vec{\nabla}_A) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \underbrace{(\vec{\nabla}\cdot\vec{\nabla})}_{\nabla^2}\vec{A}$$ **

فقط در دستگاه دکارتی برقرار است :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \cdots + \cdots \Rightarrow \nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \cdots$$

شکل: معادله موج .E.M

- پتانسیل های برداری را بکار ؟

- معادله موج ؟

قوانین ماکسول :

$$\vec{\nabla}\cdot\vec{B} = 0 \quad (1)$$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad (2)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (3)$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \quad (4) \quad ; \quad \frac{1}{c^2} = \epsilon_0\mu_0$$

تراوایی مغناطیسی

ثابت گذردهی خلأ

معادله موج در خلأ

③ $\longrightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$ ④ : $\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

$* * : \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$

② طبق رابطه : 0

$\Rightarrow -\nabla^2\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$

$\Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$

$\Rightarrow \Box\vec{E} = 0$

④ $\longrightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \Rightarrow -\nabla^2\vec{B} = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$

③ : $-\frac{\partial\vec{B}}{\partial t}$

$** : \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$; $\vec{\nabla}\cdot\vec{B} \to 0$

$\Rightarrow \Box\vec{B} = 0$

پتانسیل برداری و اسکالر

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$

چون دیورژانس هر کرل صفر است

پتانسیل برداری

③ : $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\vec{\nabla}\times\frac{\partial\vec{A}}{\partial t}$

$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\varphi$

کرل هر گرادیان صفر است

پتانسیل اسکالر

$\rightarrow \boxed{\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}}$ $*'$

مثال : $\vec{E} = \frac{q}{r^2}\hat{r} \longrightarrow \varphi = -\frac{q}{r}$

میدان ، پتانسیل

$\vec{F} = -\vec{\nabla}U$ انرژی پتانسیل $\vec{F} = q'\vec{E}$

نیرو

معادله موج با حضور source

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$: قانون گاوس

$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$: قانون آمپر $J = I/A$

$\vec{\nabla}\cdot\vec{A} = 0$: پیمانه کولن

شرط : $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$ (پیمانه لورنتس)

$\vec{B} = \vec{\nabla}\times\vec{A}$; $\vec{A}' = \vec{A} + \vec{\nabla}\chi \Rightarrow \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times\vec{\nabla}\chi} \Rightarrow \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} \Rightarrow \vec{B}' = \vec{B}$

$\vec{E} = -\vec{\nabla}\varphi + \frac{\partial\vec{A}}{\partial t}$; $\varphi' = \varphi + \frac{\partial\chi}{\partial t} \rightarrow \vec{\nabla}\varphi' = \vec{\nabla}\varphi + \frac{\partial}{\partial t}\vec{\nabla}\chi$

$\vec{E}' = -\vec{\nabla}\varphi' + \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\varphi - \cancel{\frac{\partial}{\partial t}\vec{\nabla}\chi} + \frac{\partial\vec{A}}{\partial t} + \cancel{\frac{\partial}{\partial t}\vec{\nabla}\chi} = -\vec{\nabla}\varphi + \frac{\partial\vec{A}}{\partial t} = \vec{E}$

$*' \longrightarrow \underbrace{\vec{\nabla}\cdot\vec{E}}_{\rho/\epsilon_0} = -\vec{\nabla}\cdot\vec{\nabla}\varphi - \frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\vec{A}}_{-\frac{1}{c^2}\frac{\partial\varphi}{\partial t}} \Rightarrow -\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \rho/\epsilon_0 \longrightarrow \boxed{\Box\varphi = -\rho/\epsilon_0}$

از پیمانه لورنتس : $-\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$

اگر میدان‌ها ایستا باشند $\rightarrow \frac{\partial\varphi}{\partial t} = 0$

معادله پواسون : $\nabla^2\varphi = -\rho/\epsilon_0$

قانون ارستد : $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$

$\vec{E} \xrightarrow{*'} -\vec{\nabla}\varphi - \partial\vec{A}/\partial t$

$\vec{\nabla}\cdot\vec{A}$ از پیمانه لورنتس $-\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$

$$\boxed{\Box\vec{A} = -\mu_0\vec{J}}$$

$$\Rightarrow \cancel{-\frac{1}{c^2}\vec{\nabla}\frac{\partial\varphi}{\partial t}} - \nabla^2\vec{A} + \cancel{\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\varphi} + \frac{1}{c^2}\partial^2\vec{A}/\partial t^2 = \mu_0\vec{J} \Rightarrow -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}$$